

Group Theory Assignments

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Rules

- One person per assignment.
- Let me know your preferences before November 20, 2004.
I will then fix determine your assignment by November 22, 2004.
- The assignment should be submitted to me before December 5, 2004.
- You are allowed (and encouraged) to consult me.
- Visit the library and the internet!
- Use any CA package (especially GAP and Magma can be very useful).
- If there are no satisfactory assignments in the list below, contact me with a wish for another assignments, preferably with a indication of your interests.

Assignments

- (1) Deal (succinctly) with the representation theory of the symmetric groups S_n ($n \in \mathbb{N}$). You know that conjugacy classes correspond to partitions of n . There is a way to let irreducible representations correspond to partitions of n as well, and to compute the corresponding characters ($n = 25$ is close to today's computational limit). Use this theory to determine the character table of Sym_6 . See, for instance, [M.A.A. van Leeuwen, A.M. Cohen, and B. Lissner, *LiE Manual*, manual for the software package LiE for Lie group theoretical computations, CWI/CAN, 1992] or [G.D. James & A. Kerber: *The Representation Theory of the Symmetric Group*. Encyclopedia of Mathematics, vol. 16, Springer-Verlag 1981].
- (2) The finite group $\text{SL}(2, \mathbb{F}_9)$ of all 2×2 matrices of determinant 1 with coefficients in the finite field of order 9, has order 720.
 - (a) Determine its character table.
 - (b) The group has a center of order two. With which familiar group is the quotient isomorphic? Give a proof of this result.
 - (c) In which dimension does a faithful representation occur? Can you construct it?
 - (d) If α is an automorphism of $\text{SL}(2, \mathbb{F}_9)$ and ρ a representation of this group, then $\rho \circ \alpha$ is again a representation of $\text{SL}(2, \mathbb{F}_9)$. Prove this.
 - (e) Apply the previous part to the automorphism that changes each entry of a matrix of $\text{SL}(2, \mathbb{F}_9)$ to its third power. It induces a permutation of characters occurring in the character table. Indicate which permutation.
- (3) The finite group $G = \text{SL}(2, \mathbb{F}_5)$ of all 2×2 matrices of determinant 1 with coefficients in the finite field of order 5, has order 120.
 - (a) Determine its character table.
 - (b) The group has a center of order two. With which familiar group is the quotient isomorphic? Give a proof of this result.
 - (c) Conclude from the previous items that G has a faithful irreducible representation of degree 2. Denote by ρ its character.
 - (d) For χ an irreducible character of G , write $P_\chi(t) = \sum_{i=0}^{\infty} (\chi, \rho^i) t^i$. This is in fact a rational expression in t . Try and find this expression! (Hint:

view G as a subgroup of $SU(2)$ and consider the restrictions to G of the irreducible characters of the latter group described in the notes.)

- (4) Let G be a finite group, with a permutation representation $\pi : G \rightarrow S_n$ of degree n .
- Prove that $(1, \pi)$ is the number of orbits of G on $X = \{1, \dots, n\}$.
 - Suppose that π is transitive (that is, $\pi(G)$ has only one orbit on X). Show that (π, π) is the number of orbits of G on $X \times X$ (via the diagonal action).
 - An orbit of G on $X \times X$ is called an *orbital*. So, an orbital is a relation on X . Suppose E is a symmetric orbital. Prove that $\pi(G)$ is a group of automorphisms of the graph whose vertex set is X and in which $x, y \in X$ are adjacent vertices iff $(x, y) \in E$.
 - Show that every graph (X, E) whose automorphism group is transitive on X as well as on E can be constructed as in the previous item.
 - Give a two parameter series of examples by letting X be the collection of sets of size k from a set of size d , and $G = S_d$. The best examples are obtained by letting adjacency be ‘having intersection of size $k - 1$ ’. Show that, with this choice of adjacency, the orbitals are precisely the relations ‘having distance i in the graph’.
 - Analyse the characters involved in the permutation character in the examples of the previous item (as far as you can, possibly explicitly for small d).
- (5) Treat the induced representations. Let G be a group with subgroup H , and suppose that a representation $\rho : H \rightarrow GL(V)$ is given.
- Describe the character of the induced representation ρ^G of G .
 - Prove that, if χ denotes the character of ρ , and χ^G is the character of ρ^G , while ψ is an arbitrary character of G , we have: $(\psi, \chi^G) = (\psi|_H, \chi)$, where $|_H$ denotes the restriction (of a function) to H .
 - Verify that all irreducible representations of the dihedral groups are induced from linear (that is, 1-dimensional) representations.
 - Check which irreducible representations of Alt_6 are induced from proper subgroups.
 - Can you also determine which irreducible representations of Alt_6 occur in induced representations from proper subgroups?
- (6) Consider the character table below ... that is, I claim it is a character table of a finite group. Find all groups G up to isomorphism, of which this table is a character table. All properties of the group that you can derive from the table are welcome. Record all your arguments.

G	e	a	b	c	d	f	g	h	i	j	k	l	m
$ \text{ConjClass}_i $	1	6	1	12	24	12	6	6	12	32	32	24	24
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	-1	1	1	1	1	1	-1	-1
χ_3	2	2	2	0	0	0	2	2	2	-1	-1	0	0
χ_4	3	-1	3	-1	1	-1	3	-1	-1	0	0	-1	1
χ_5	3	-1	3	-1	1	-1	-1	3	-1	0	0	1	-1
χ_6	3	3	3	-1	-1	-1	-1	-1	-1	0	0	1	1
χ_7	3	-1	3	1	-1	1	3	-1	-1	0	0	1	-1
χ_8	3	-1	3	1	-1	1	-1	3	-1	0	0	-1	1
χ_9	3	3	3	1	1	1	-1	-1	-1	0	0	-1	-1
χ_{10}	4	0	-4	2	0	-2	0	0	0	1	-1	0	0
χ_{11}	4	0	-4	-2	0	2	0	0	0	1	-1	0	0
χ_{12}	6	-2	6	0	0	0	-2	-2	2	0	0	0	0
χ_{13}	8	0	-8	0	0	0	0	0	0	-1	1	0	0

Hint: a character table records the information

$$a_{ijk} = \#\{z \in \text{ConjClass}_i \mid \exists y \in \text{ConjClass}_j \exists z \in \text{ConjClass}_k \quad z = xy\}.$$

(Consult literature.) Besides, the character table shows that the group has a faithful representation of degree 4. Character χ_3 shows that there is a normal subgroup of order 8. Determine its structure, and start your ‘reconstruction’ of the faithful 4-dimensional representation with this subgroup.