Root filtration spaces from Lie algebras and abstract root groups [1]

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Abstract

Both Timmesfeld's abstract root subgroups and simple Lie algebras generated by extremal elements lead to root filtration spaces: synthetically defined geometries on points and lines which can be characterized as root shadow spaces of buildings. Here we show how to obtain the root filtration space axioms from root subgroups and classical Lie algebras.

Key words: Abstract root groups, buildings, groups of Lie type, Lie algebras, shadow spaces

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1 Introduction

In this paper, we study partial linear spaces that are root shadow spaces of spherical buildings. This means that the Coxeter type of the building comes from a Dynkin diagram and that the shadow space has points that are flags whose types are the nodes adjacent to the node extending the Dynkin diagram to an affine diagram (see Definition 3 below). The only irreducible diagram in which this is more than one node is A_n; the points of the corresponding root shadow space are the incident point-hyperplane pairs of a projective geometry of rank n, a building of type A_n. We provide an axiom system in terms of points, lines, and relations on the point set. A point-line space satisfying these

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axioms will be called a root filtration space. Elsewhere [3], we show that these root filtration spaces are in fact root shadow spaces of buildings.

The main purpose of this paper is to show that both groups generated by abstract root subgroups as studied by Timmesfeld and Lie algebras generated by extremal elements give rise to root filtration spaces. Here an element x of a Lie algebra L over a field k of odd characteristic is called extremal if $[x,[x,L]] \subseteq kx$ (see Definition 14 for the general case). The long root elements of classical Lie algebras are examples. Thus, together with [3] we find an alternative proof of a major part of Timmesfeld's classification on abstract root subgroups of simple groups as well as of parts of the determination of classical simple Lie algebras generated by extremal elements.

We proceed as follows. In section 2, we introduce the notion of root filtration space. In comparison with other axiom systems regarding point-line geometries known to us, the notion of a filtration around a point is new; the idea stems from a Lie algebra filtration described in Corollary 23.

In section 3, it is shown that Timmesfeld's non-degenerate sets of abstract root subgroups are related to non-degenerate root filtration spaces (Theorem 13). In Section 4 we deal with Lie algebras L generated by extremal elements. We derive some properties of the set $\mathcal{E}$ of projective points corresponding to nonzero extremal elements of L and recall some of those found in [4]. These properties lead to the structure of a root filtration space on $\mathcal{E}$ (Theorem 28). Exponentiation turns the elements of $\mathcal{E}$ into abstract root subgroups.

2 Root filtration spaces

We introduce root filtration spaces and derive some of their properties. We begin with some notation for relations on a set $\mathcal{E}$. Let $x \in \mathcal{E}$. For a relation $\mathcal{X}$ on $\mathcal{E}$, we denote by $\mathcal{X}(x)$ the set of all elements $y \in \mathcal{E}$ with $(x,y) \in \mathcal{X}$. If, in addition, $y, z \in \mathcal{E}$ and $Y \subseteq \mathcal{E}$, we write $\mathcal{X}(x,y)$ for $\mathcal{X}(x) \cap \mathcal{X}(y)$, $\mathcal{X}(x,y,z)$ for $\mathcal{X}(x) \cap \mathcal{X}(y) \cap \mathcal{X}(z)$, and $\mathcal{X}(Y)$ for $\bigcap_{y \in Y} \mathcal{X}(y)$, etc.

A *point-line space* (or just *space*) is a pair $(\mathcal{E}, \mathcal{F})$ consisting of a set $\mathcal{E}$ (of points) and a collection $\mathcal{F}$ of subsets of $\mathcal{E}$ of size at least 2 (whose members are called lines). A space is called a *gamma space* if, for each point p and each line l not on p, the set of points on l collinear with p is either empty, a singleton, or all of l. It is called a *partial linear space* if every pair of distinct points is on at most one line. A *subspace* of $(\mathcal{E}, \mathcal{F})$ is a subset of $\mathcal{E}$ containing each line that has at least two points in common with it. The *rank* of a linear space is the length of a maximal chain of proper non-trivial subspaces; if there is no such chain, the rank is said to be ∞. A *singular subspace* of a space is a subspace in which any two points are collinear. The *singular rank* of a space is the supremum of all ranks of maximal singular subspaces.

We say that a subspace of a point-line space is a *hyperplane* if every line has a non-empty intersection with it. Thus, the whole point set is a hyperplane.

The definitions of *polar space*, *non-degeneracy* of a polar space, and *rank* of a polar space, are as in [2].

Let $(\mathcal{E},\mathcal{F})$ be a partial linear space. For $\{\mathcal{E}_i\}_{-2\le i\le 2}$, a quintuple of disjoint symmetric relations partitioning $\mathcal{E}\times\mathcal{E}$, we call $(\mathcal{E},\mathcal{F})$ a root filtration space with filtration $\{\mathcal{E}_i\}_{-2\le i\le 2}$ if the following properties are satisfied, where we write $\mathcal{E}_{\le i}$ for $\cup_{j\le i}\mathcal{E}_j$.

(A) The relation $\mathcal{E}_{-2}$ is equality on $\mathcal{E}$.
(B) The relation $\mathcal{E}_{-1}$ is collinearity of distinct points of $\mathcal{E}$.
(C) There is a map $\mathcal{E}_1\to\mathcal{E}$, denoted by $(u,v)\mapsto[u,v]$ such that, if $(u,v)\in\mathcal{E}_1$ and $x\in\mathcal{E}_i(u)\cap\mathcal{E}_j(v)$, then $[u,v]\in\mathcal{E}_{\le i+j}(x)$.
(D) For each $(x,y)\in\mathcal{E}_2$, we have $\mathcal{E}_{\le 0}(x)\cap\mathcal{E}_{\le -1}(y)=\emptyset$.
(E) For each $x\in\mathcal{E}$, the subsets $\mathcal{E}_{\le -1}(x)$ and $\mathcal{E}_{\le 0}(x)$ are subspaces of $(\mathcal{E},\mathcal{F})$.
(F) For each $x\in\mathcal{E}$, the subset $\mathcal{E}_{\le 1}(x)$ is a hyperplane of $(\mathcal{E},\mathcal{F})$.

In arguments, condition (C) applied to x will be referred as *filtration around* x. According to Lemma 1(ii) below, $[u,v]$ is the unique point in $\mathcal{E}_{\le -1}(u)\cap\mathcal{E}_{\le -1}(v)$, so the map $[\cdot,\cdot]$ is uniquely determined by the relations $(\mathcal{E}_i)_i$.

We adopt the terminology of [7], referring to Condition (D) as the *triangle condition on* x, y, z. It is equivalent to the seemingly more general statement that, for each $(x,y)\in\mathcal{E}_2$, we have $\mathcal{E}_{\le i}(x)\cap\mathcal{E}_{\le j}(y)=\emptyset$ whenever $i+j<0$.

Condition (E) can be replaced by the statement that $\mathcal{E}_{\le -i}(x)$ is a subspace of $(\mathcal{E},\mathcal{F})$ for each i (see Lemma 1(i) below).

We call a pair $(x,z)\in\mathcal{E}_i$ *hyperbolic* if $i=2$, *special* if $i=1$, *polar* if $i=0$, *collinear* if $i=-1$ (so collinearity is only used for distinct points), and *commuting* (notation $[x,z]=0$) if $i\le 0$.

Lemma 1 *In a root filtration space* $(\mathcal{E},\mathcal{F})$ *the following properties hold.*

(i) For each $i\in\{-2,\dots,2\}$ *and each* $x\in\mathcal{E}$*, the subset* $\mathcal{E}_{\le i}(x)$ *is a subspace of* $(\mathcal{E},\mathcal{F})$*.*
(ii) If $(u,v)\in\mathcal{E}_1$*, then* $[u,v]$ *is the unique common neighbor of both* u *and* v *in the collinearity graph* $(\mathcal{E},\mathcal{E}_{-1})$ *of* $(\mathcal{E},\mathcal{F})$*.*
(iii) If $(u,v)\in\mathcal{E}_1$*, then* $\mathcal{E}_0(u)\cap\mathcal{E}_2(v)\subseteq\mathcal{E}_1([u,v])$*.*
(iv) If $(x,y)\in\mathcal{E}_0$ *and* $z\in\mathcal{E}_{-1}(y)$*, then either* $z\in\mathcal{E}_{\le 0}(x)$*, or* $z\in\mathcal{E}_1(x)$ *and* $\mathcal{E}_{-1}(x,y,z)=\{[x,z]\}$*.*

- *(v)* *If* (x,q) *and* (u,z) *belong to* $\mathcal{E}_1$ *whereas* $u = [x,q]$ *and* $q = [u,z]$, *then* $(x,z) \in \mathcal{E}_2$.
- *(vi)* *If* P *is a pentagon in the collinearity graph* $(\mathcal{E}, \mathcal{E}_{-1})$ *(that is, the induced subgraph is a pentagon), then each distinct non-collinear pair of points of* P *is polar.*
- *(vii)* *If* $(u,v) \in \mathcal{E}_1$, *then* $\mathcal{E}_{-1}(u) \cap \mathcal{E}_0([u,v]) \subseteq \mathcal{E}_1(v)$.

Proof. (i). This is stated in (E) for $i = -1, 0$. It is trivial for $i = -2$ and $i = 2$ since singletons and the whole set $\mathcal{E}$ are subspaces. Since a hyperplane is a subspace, it follows from (F) for $i = 1$.

(ii). By filtration around u, we have $[u,v] \in \mathcal{E}_{\leq -1}(u)$, and similarly $[u,v] \in \mathcal{E}_{\leq -1}(v)$. Since, by the disjointness assumption on $\mathcal{E}_1$ and $\mathcal{E}_{-1}$, the points u and v are not collinear, $[u,v]$ cannot coincide with u or v. Suppose now that y is a point collinear with both u and v. Then, by (C) we have $[u,v] \in \mathcal{E}_{\leq -2}(y)$, which implies $y = [u,v]$ by (A).

(iii). By (ii), $[u,v] \in \mathcal{E}_{-1}(u,v)$. Let $x \in \mathcal{E}_0(u) \cap \mathcal{E}_2(v)$. Applying (D) to x and $[u,v]$ we find $[u,v] \notin \mathcal{E}_2(x)$ and applying (D) to x and v, we find $[u,v] \notin \mathcal{E}_{\leq 0}(x)$. Therefore, $x \in \mathcal{E}_1([u,v])$.

(iv). By (D), $z \in \mathcal{E}_{\leq 1}(x)$. Suppose $z \in \mathcal{E}_1(x)$, so $[x,z]$ is a point collinear with x and z. Then, by (C) applied to y, we find $[x,z] \in \mathcal{E}_{-1}(y)$, so by (B) the point $[x,z]$ is as required. Finally, the only point of $\mathcal{E}_{\leq -1}(z,x)$ is $[z,x]$.

(v). Observe that x, u, q, z is a path in the collinearity graph of $(\mathcal{E}, \mathcal{F})$. If $(x,z) \in \mathcal{E}_{\leq 1}$, then by the filtration around x, we find $q = [u,z] \in \mathcal{E}_{\leq 0}(x)$, contradicting the assumption $(q,x) \in \mathcal{E}_1$. Hence, $(x,z) \in \mathcal{E}_2$.

(vi). Let a, b, c, d, e be the points of P, chosen so that $\{e, a\}$ and successive pairs of points are collinear. By (v) and the fact that a pair of hyperbolic points has mutual distance at least 3 (possibly ∞) in the collinearity graph (a consequence of (D)), no two consecutive non-collinear pairs of the pentagon can be special. So at most two non-collinear pairs can be special. Therefore, after a suitable renaming of the points of P, the pairs (a,c) and (a,d) may be assumed not special, whence polar. Now $b, e \in \mathcal{E}_{-1}(a)$ and $c, d \in \mathcal{E}_0(a)$, so P is contained in $\mathcal{E}_{\leq 0}(a)$. If $(b,d) \in \mathcal{E}_1$, then, by (ii) and (iv), the point $c = [b,d]$ is collinear with a, contradicting that P is a pentagon. Therefore $(b,d) \in \mathcal{E}_0$. Similarly, it can be shown that $(a,d) \in \mathcal{E}_0$ and, finally, that $(b,e) \in \mathcal{E}_0$.

(vii). Let $y \in \mathcal{E}_{-1}(u) \cap \mathcal{E}_0([u,v])$. In view of the triangle condition on $[u,v]$, y, v, we must have $y \in \mathcal{E}_{\leq 1}(v)$. If $[y,v] = 0$ then $(y, [u,v]) \in \mathcal{E}_{-1}$ by the filtration around y, a contradiction. □

Examples 2 Here are some examples of root filtration spaces.

(i). Every linear space is a trivial example of a root filtration space with $\mathcal{E}_i = \emptyset$ for $i \geq 0$.

(ii). Every space without lines is a trivial example of a root filtration space with $\mathcal{E}_i = \emptyset$ for $-2 < i < 2$ and $\mathcal{E}_2$ the relation of being distinct. Even if we keep $\mathcal{E}_1 = \mathcal{E}_{-1} = \emptyset$ and allow for $\mathcal{E}_0 \neq \emptyset$, the result is a root filtration space. For example, if $(\mathcal{E}, \mathcal{L})$ is a polar space, taking $\mathcal{E}_2$ to be the non-collinearity relation for distinct points and $\mathcal{E}_0$ the complement in $\mathcal{E} \times \mathcal{E}$ of $\mathcal{E}_{-2} \cup \mathcal{E}_2$, we obtain a root filtration space $(\mathcal{E}, \emptyset)$ with $\mathcal{E}_{-1} = \mathcal{E}_1 = \emptyset$.

(iii). Every generalized hexagon $(\mathcal{E}, \mathcal{F})$ with $\mathcal{E}_i$ for $i = 1, 2$ the set of points at mutual distance $i + 1$, and $[x, y]$ the unique point collinear with both x and y for $(x, y) \in \mathcal{E}_1$, is a root filtration space with $\mathcal{E}_0 = \emptyset$.

(iv). Let $\mathbb{P}$ be a projective space and let $\mathbb{H}$ be a collection of hyperplanes forming a subspace of the dual of $\mathbb{P}$ annihilating $\mathbb{P}$. The latter means that the intersection of all hyperplanes of $\mathbb{H}$ is empty. If $\mathbb{P}$ has finite rank, this condition forces $\mathbb{H}$ to be the dual of $\mathbb{P}$. Take $\mathcal{E}$ to be the set of incident pairs from $\mathbb{P} \times \mathbb{H}$. The set $\mathcal{F}$ of lines is built up of two kinds: those consisting of all (x, H) with hyperplane $H \in \mathbb{H}$ fixed and x running through the points of a line of $\mathbb{P}$ inside H, and those consisting of all (x, H) with point x fixed and H running through the hyperplanes in $\mathbb{H}$ containing a fixed codimension 2 subspace of $\mathbb{P}$ containing x. So, $((x, H), (y, K)) \in \mathcal{E}_{-1}$ iff $x = y$ or $H = K$ (but not both). Then $(\mathcal{E}, \mathcal{F})$ is a root filtration space with $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$ defined as follows: $(x, H) \in \mathcal{E}_0((y, K))$ iff $x \in K$ and $y \in H$ and $x \neq y$ and $H \neq K$, $(x, H) \in \mathcal{E}_1((y, K))$ iff $x \in K$ (in which case $[(x, H), (y, K)] = (x, K)$) or $y \in H$ (in which case $[(x, H), (y, K)] = (y, H)$) but not both, and $(x, H) \in \mathcal{E}_2((y, K))$ iff $x \notin K$ and $y \notin H$. We denote this root filtration space by $\mathcal{E}(\mathbb{P}, \mathbb{H})$.

(v). Let $(\mathcal{P}, \mathcal{E})$ be a non-degenerate polar space. Then the Grassmann space $(\mathcal{E}, \mathcal{F})$ on $(\mathcal{P}, \mathcal{E})$, where $\mathcal{F}$ consists of the pencils of lines on a point in a singular plane, is a root filtration space with $l \in \mathcal{E}_{-1}(m)$ iff l and m span a singular plane, $l \in \mathcal{E}_0(m)$ iff either l and m span a singular subspace not contained in a plane, or l and m intersect but do not span a singular plane, $l \in \mathcal{E}_1(m)$ iff there is a unique line n such that both the span of l and n and the span of n and m are singular planes (in which case $n = [l, m]$, and the span of l and m is not a singular subspace), and $\mathcal{E}_2$ is the complement of $\mathcal{E}_{\leq 1}$ in $\mathcal{E} \times \mathcal{E}$. The same construction for a projective space instead of a polar space leads to a root filtration space with $\mathcal{E}_1 = \mathcal{E}_2 = \emptyset$.

Let $\mathbb{P}$ and $\mathbb{H}$ be as in (iv). Consider the space $(\mathcal{P}, \mathcal{L})$ whose point set $\mathcal{P}$ is the disjoint union of $\mathbb{P}$ and $\mathbb{H}$ and whose line set $\mathcal{L}$ is the union of the line set of $\mathbb{P}$, the line set of $\mathbb{H}$ and the set of all unordered pairs $\{x, H\}$ with $x \in \mathbb{P}$ and $H \in \mathbb{H}$ such that $x \in H$. This is a non-degenerate polar space, called the *dualized projective space* of $\mathbb{P}$ and $\mathbb{H}$. The root filtration space $\mathcal{E}(\mathbb{P}, \mathbb{H})$ defined in (iv) is a subspace of the Grassmann space on the dualized projective space $(\mathcal{P}, \mathcal{L})$.

(vi). Suppose that $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)})$ and $(\mathcal{E}^{(2)}, \mathcal{F}^{(2)})$ are root filtration spaces. Let $\mathcal{E}$ be the disjoint union of $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ and let $\mathcal{F}$ be the disjoint union of $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$. Then $(\mathcal{E}, \mathcal{F})$ is a root filtration space with filtration $\mathcal{E}_0 = \mathcal{E}_0^{(1)} \cup \mathcal{E}_0^{(2)} \cup (\mathcal{E}^{(1)} \times \mathcal{E}^{(2)}) \cup (\mathcal{E}^{(2)} \times \mathcal{E}^{(1)})$ and $\mathcal{E}_i = \mathcal{E}_i^{(1)} \cup \mathcal{E}_i^{(2)}$ for $i \neq 0$.

Each spherical building whose Coxeter diagram comes from a Dynkin diagram corresponds to a root filtration space. This holds in particular for all thick spherical buildings. To clarify this, we need the following definitions. Recall that a Dynkin diagram is a Coxeter diagram of a Weyl group whose bonds with an even label greater than 2 are directed.

Definition 3 Let Y_n be an irreducible Dynkin diagram of rank $n > 1$. We number its nodes with $1, \ldots, n$ as in Bourbaki [1]. Denote by X_n the corresponding Coxeter diagram (obtained by removing the arrow of a multiple bond). To avoid confusion between the Dynkin diagram and the Coxeter diagram, we shall write $X_n = (\mathrm{B}|\mathrm{C})_n$ for the Coxeter diagram corresponding to both B_n and C_n.

Let $\widetilde{Y}_n$ be the extended (or affine) Dynkin diagram of Y_n. Its nodes are those of Y_n and an additional node, numbered 0. By J we denote the subset of $\{1, \ldots, n\}$ consisting of all nodes of Y_n adjacent to 0. Then $J = \{1, n\}$ if $M = \mathrm{A}_n$, and $J = \{j\}$, where $j = 2$ if $Y_n = \mathrm{B}_n$, $j = 1$ if $Y_n = \mathrm{C}_n$, $j = 2$ if $Y_n = \mathrm{D}_n$ or E_6, $j = 1$ if $Y_n = \mathrm{E}_7$, $j = 8$ if $Y_n = \mathrm{E}_8$, $j = 1$ if $Y_n = \mathrm{F}_4$, $j = 2$ if $Y_n = \mathrm{G}_2$. We shall call J the *root nodes* or, if appropriate, j the *root node* of Y_n.

Let $\mathcal{C}$ be a building of type X_n. Following [8], we view it as a chamber system over $R = \{1, \ldots, n\}$. Let J be an arbitrary subset of R. The *J-shadow* of a chamber c of $\mathcal{C}$ is the $(R \setminus J)$-cell containing c. For $j \in J$, we define a *j-line* to be the union of all $(R \setminus J)$-cells containing a chamber from a given j-panel. The pair $(\mathcal{E}, \mathcal{F})$ consisting of the set $\mathcal{E}$ of all J-shadows and the set $\mathcal{F}$ of all j-lines, for $j \in J$, is called the *shadow space* for $\mathcal{C}$ of type $X_{n,J}$. If $J = \{j\}$, we also write $X_{n,j}$ instead of $X_{n,J}$. If J is the set of root nodes of a corresponding Dynkin diagram Y_n, we call $(\mathcal{E}, \mathcal{F})$ the *root shadow space* of $\mathcal{C}$ with respect to Y_n. If X_n has multiple bonds, there are two choices for Y_n, whence two root shadow spaces of $\mathcal{C}$.

Of Examples 2, (iii) represents the root shadow space of a building of type G_2, its dual corresponds to the other interpretation of G_2 as a Dynkin diagram. Choice (iv) represents A_n, (v) represents both Dynkin types B_n and D_n, and the polar spaces of (ii) represent the Dynkin type C_n. In [3], we prove that root shadow spaces are indeed root filtration spaces.

We next focus on a non-degeneracy property for root filtration spaces.

Lemma 4 *The following three conditions for a root filtration space* $(\mathcal{E}, \mathcal{F})$ *with respect to* $(\mathcal{E}_i)_i$ *are equivalent.*

(i) For each $(x, u) \in \mathcal{E}_{-1}$ *there exists a point in* $\mathcal{E}_2(x) \cap \mathcal{E}_1(u)$*.*
(ii) For each $(x, u) \in \mathcal{E}_{-1}$ *there exists a point in* $\mathcal{E}_{-1}(u) \cap \mathcal{E}_1(x)$*.*
(iii) For each $x \in \mathcal{E}$ *with* $\mathcal{E}_{-1}(x) \neq \emptyset$*, there exists a point in* $\mathcal{E}_2(x)$*.*

Proof. (i)⇒(ii). Let $(x, u) \in \mathcal{E}_{-1}$. Now take $z \in \mathcal{E}_2(x) \cap \mathcal{E}_1(u)$ as in (i). By Lemma 1(ii), $v = [u, z]$ is collinear with u and by filtration around x it belongs to $\mathcal{E}_{\leq 1}(x)$. But $v \in \mathcal{E}_{\leq 0}(x)$ would contradict (D), the triangle condition on x, v, and z. Therefore, $v \in \mathcal{E}_1(x)$.

(ii)⇒(iii). Assume $(x, u) \in \mathcal{E}_{-1}$. By (ii), there are $q \in \mathcal{E}_{-1}(u) \cap \mathcal{E}_1(x)$ and $z \in \mathcal{E}_{-1}(q) \cap \mathcal{E}_1(u)$. By Lemma 1(v), $z \in \mathcal{E}_2(x)$.

(iii)⇒(i). Let $(x, u) \in \mathcal{E}_{-1}$. Take $a \in \mathcal{E}_2(x)$. As $\mathcal{E}_{\leq 1}(a)$ is a hyperplane there is a unique point y of $\mathcal{E}_{\leq 1}(a)$ on the line xu. We are done if $y = u$. Therefore we assume that $u \in \mathcal{E}_2(a)$. As in the proof of implication (i)⇒(ii) we see that $y \in \mathcal{E}_1(a)$ and for $v' = [a, y]$ we have $v' \in \mathcal{E}_1(u) \cap \mathcal{E}_{-1}(y)$. Let $w \in \mathcal{E}_2(y)$. Then there is a unique point v of $\mathcal{E}_{\leq 1}(w)$ on the line $v'y$. Observe that $(v, u) \in \mathcal{E}_1$. Again, $(v, w) \in \mathcal{E}_1$ and for $z' = [v, w]$ we have $z' \in \mathcal{E}_1(y)$. Together with the previous observation, by Lemma 1(v), this implies $(z', u) \in \mathcal{E}_2$. We also have $z' \in \mathcal{E}_{-1}(w)$ and hence there is a unique point $z \neq z'$ on the line $z'w$ which is in $\mathcal{E}_{\leq 1}(u)$. As $w \in \mathcal{E}_2(y)$ the point z' is the unique point of $z'w \cap \mathcal{E}_{\leq 1}(y)$, so $z \in \mathcal{E}_2(y)$. This implies that u is the unique point of $\mathcal{E}_{\leq 1}(z)$ on the line $xy = xu$. In particular, as $x \neq u$, we have $x \in \mathcal{E}_2(z)$. We found $z \in \mathcal{E}_2(x) \cap \mathcal{E}_1(u)$, as required. □

The second condition means that the local space (as defined in the text following Theorem 6.1 of [2]) at every point is non-degenerate. The third condition means that the hyperplane $\mathcal{E}_{\leq 1}(x)$ is proper.

For a further discussion of non-degeneracy, we recall that a line is said to be thick if it has at least three points, and that a point-line space is thick if each of its lines is thick.

Lemma 5 *Assume that* $(\mathcal{E}, \mathcal{F})$ *is a root filtration space with respect to* $(\mathcal{E}_i)_i$ *in which the conditions of Lemma 4 hold.*

(i) If $(\mathcal{E}, \mathcal{F})$ *is thick and* $(\mathcal{E}, \mathcal{E}_{-1})$ *is connected, then so is* $(\mathcal{E}, \mathcal{E}_2)$*.*
(ii) If $(\mathcal{E}, \mathcal{E}_2)$ *is connected and* $\mathcal{F}$ *is non-empty, then* $(\mathcal{E}, \mathcal{E}_{-1})$ *is connected.*

Proof. (i). Let $(x, y) \in \mathcal{E}_{-1}$. Since the line $xy \in \mathcal{F}$ on x and y is thick, it has a third point u. Now take $z \in \mathcal{E}_2(x) \cap \mathcal{E}_1(u)$. Then, by (F), $z \in \mathcal{E}_2(y)$, so x and

y have distance 2 in the graph $(\mathcal{E}, \mathcal{E}_2)$.

(ii). Suppose $l \in \mathcal{F}$, $z \in l$, and $x \in \mathcal{E}_2(z)$. By (F), the line l has a point $y \in \mathcal{E}_1(x)$. In view of Lemma 1(ii), $x[x, y]$ is a line on x and the distance between x and z in $(\mathcal{E}, \mathcal{E}_{-1})$ is at most 3. In particular, x lies on a line and we can finish by induction on the distance to a point of l in $(\mathcal{E}, \mathcal{E}_2)$. $\Box$

The thin case of Example 2(iii), the ordinary hexagon, shows that the thickness condition is necessary in Lemma 5(i).

Examples 2(i), (ii), (vi) illustrate that the notion of a root filtration space is too general for a classification. Therefore, we shall impose restrictions like those of Lemmas 4 and 5. We call a root filtration space $(\mathcal{E}, \mathcal{F})$ with filtration $(\mathcal{E}_i)_i$ *non-degenerate* if the following two conditions are satisfied.

(G) For each $x \in \mathcal{E}$ the set $\mathcal{E}_2(x)$ is not empty.
(H) The graph $(\mathcal{E}, \mathcal{E}_{-1})$ is connected.

Lemma 6 *Suppose* $(u, v) \in \mathcal{E}_1$ *and* $y \in \mathcal{E}_0([u, v], u, v)$. *If* $\mathcal{E}_{-1}(v, y) \neq \emptyset$, *then* $\mathcal{E}_{-1}([u, v], y) \neq \emptyset$.

Proof. Put $x = [u, v]$. Suppose $\mathcal{E}_{-1}(x, y) = \emptyset$ and $w \in \mathcal{E}_{-1}(v, y)$. Then $w \notin \mathcal{E}_{\leq -1}(x)$. By the triangle condition for u, y, w, we must have $u \in \mathcal{E}_{\leq 1}(w)$. Assume that $u \in \mathcal{E}_{\leq 0}(w)$. Then the filtration around w gives $x = [u, v] \in \mathcal{E}_{\leq -1}(w)$, whence $w \in \mathcal{E}_{\leq -1}(x, y)$, contradicting $w \notin \mathcal{E}_{\leq -1}(x)$. Consequently, $(u, w) \in \mathcal{E}_1$, so, by Lemma 1(vi), the 5-circuit $x, u, [u, w], w, v$ cannot be a pentagon. This forces $[u, w] \in \mathcal{E}_{\leq -1}(x)$ (observe that $[u, w] \notin \mathcal{E}_{\leq -1}(v)$ for otherwise $[u, w] = [u, v] = x$ would be collinear with w) and, in view of the filtration around y, also $[u, w] \in \mathcal{E}_{\leq -1}(y)$, a contradiction. $\Box$

We show that polar pairs have distance 2 in the collinearity graph of a nondegenerate root filtration space. The nondegeneracy is needed in view of Example 2(vi).

Lemma 7 *Assume that* $(\mathcal{E}, \mathcal{F})$ *is a root filtration space satisfying (G). Then* $(\mathcal{E}, \mathcal{F})$ *is the disjoint union of connected subspaces* $\mathcal{B}_i$ *such that* $\mathcal{B}_i \times \mathcal{B}_j \subseteq \mathcal{E}_0$ *whenever* $i \neq j$ *unless* $\mathcal{B}_i \times \mathcal{B}_j \subseteq \mathcal{E}_2$, *in which case* $\mathcal{B}_i$ *and* $\mathcal{B}_j$ *are singletons. Moreover, if* $x, y \in \mathcal{B}_i$ *for some* i *and* $(x, y) \in \mathcal{E}_0$ *then* $\mathcal{E}_{-1}(x, y) \neq \emptyset$.

Proof. We first show that $\mathcal{E}_1(x) \cup \mathcal{E}_2(x)$ is contained in the connected component of x in $(\mathcal{E}, \mathcal{E}_{-1})$, except possibly when there is no line on x. If $z \in \mathcal{E}_1(x)$, then there is the path $x, [x, z], z$. If $z \in \mathcal{E}_2(x)$, and l is a line on x then there is a path $x, v, [v, z], z$ where $\{v\} = l \cap \mathcal{E}_{\leq 1}(z)$. Therefore, either z is connected to x by a path in $(\mathcal{E}, \mathcal{E}_{-1})$ or $\{x\}$ is a connected component in $(\mathcal{E}, \mathcal{E}_{-1})$ (in which case the same argument can be applied with the roles of z and x interchanged).

Suppose $(x, y) \in \mathcal{E}_0$ and $\mathcal{E}_{-1}(x, y) = \emptyset$. We show that x and y lie in different components of $(\mathcal{E}, \mathcal{E}_{-1})$. Let $v \in \mathcal{E}_{-1}(x)$. Then $v \in \mathcal{E}_{\leq 0}(y)$ by the triangle condition and Lemma 1(iv). By Lemma 4, there exists $u \in \mathcal{E}$ such that $(u, v) \in \mathcal{E}_1$ and $x = [u, v]$. By the argument for v applied to u, we also have $(u, y) \in \mathcal{E}_0$. Lemma 6 gives $\mathcal{E}_{\leq -1}(y, v) = \emptyset$. So the pair (v, y) inherits the property of having no common collinear points from the pair (x, y). Since v was chosen to be an arbitrary point collinear with x, we find that $(v, y) \in \mathcal{E}_0$ and $\mathcal{E}_{-1}(v, y) = \emptyset$ for all points in the connected component of x in $(\mathcal{E}, \mathcal{E}_{-1})$. Varying y in the same way, we find the required assertion. □

Lemma 8 *Suppose that $(\mathcal{E}, \mathcal{F})$ is non-degenerate. Assume $(x, y) \in \mathcal{E}_0$ and $u \in \mathcal{E}_{\leq -1}(x, y)$. Then there exists $v \in \mathcal{E}_{\leq -1}(x, y)$ such that v is not collinear with u. In particular, every polar pair (x, y) is contained in a quadrangle.*

Proof. By Lemma 4, there exists $y' \in \mathcal{E}_{-1}(y) \cap \mathcal{E}_1(u)$. Then $y = [u, y']$ and Lemma 1(vii) gives $y' \in \mathcal{E}_1(x)$. Set $v = [x, y']$. Then $v \in \mathcal{E}_{-1}(x)$. Furthermore, because of the filtration around y, we also have $v = [x, y'] \in \mathcal{E}_{\leq -1}(y)$. Thus $v \in \mathcal{E}_{\leq -1}(x, y)$. Also, $y' \in \mathcal{E}_1(u)$ and $y' \in \mathcal{E}_{-1}(v)$ exclude the possibility of $v \in \mathcal{E}_{\leq -1}(u)$. This proves the first assertion. The second one follows from Lemma 7. □

By Lemma 8, the relations $(\mathcal{E}_i)_i$ and the map $[\cdot, \cdot] : \mathcal{E}_1 \to \mathcal{E}$ of a non-degenerate root filtration space $(\mathcal{E}, \mathcal{F})$ are fully determined by the space $(\mathcal{E}, \mathcal{F})$ itself. For, $\mathcal{E}_{-1}$, $\mathcal{E}_0 \cup \mathcal{E}_1$, and $\mathcal{E}_2$ are the relations of having distance 1, 2, and 3 in the collinearity graph of $(\mathcal{E}, \mathcal{F})$, and, for x, y at mutual distance 2, we have $x \in E_1(y)$ if and only if x and y have a unique common neighbor (which coincides with $[x, y]$). Therefore, we will often not mention the filtration explicitly when introducing a non-degenerate root filtration space.

We are in a position to state the characterization which will be proved in [3].

Theorem 9 *Let $(\mathcal{E}, \mathcal{F})$ be a non-degenerate root filtration space. If the singular rank of $(\mathcal{E}, \mathcal{F})$ is finite, then $(\mathcal{E}, \mathcal{F})$ is isomorphic to a shadow space of type $\mathrm{A}_{n,\{1,n\}}$ $(n \geq 2)$, $(\mathrm{B|C})_{n,2}$ $(n \geq 3)$, $\mathrm{D}_{n,2}$ $(n \geq 4)$, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$, $\mathrm{F}_{4,1}$, or $\mathrm{G}_{2,2}$.*

3 Abstract root subgroups

We show that Timmesfeld's non-degenerate sets of abstract root subgroups are in fact non-degenerate root filtration spaces. We begin by recalling the notion of abstract root subgroups, appearing in Definition (1.1), Chapter II of [7] (the wording is adjusted to our setting). If A and B are subgroups of a given group G, then $[A, B]$ stands for the subgroup of G generated by all commutators $[a, b] := a^{-1}b^{-1}ab = a^{-1}a^b$ with $a \in A$ and $b \in B$. Similarly for $[a, B]$ and $[A, b]$.

Definition 10 Let G be a group. A set $\mathcal{E}$ of abelian non-trivial subgroups of G is called a set of *abstract root subgroups* of G if it satisfies the following two conditions.

(I) $G = \langle \mathcal{E} \rangle$ and $\mathcal{E}^g \subseteq \mathcal{E}$ for each $g \in G$.

(II) For each pair $a, b \in \mathcal{E}$ one of the following cases occurs, where $X = \langle a, b \rangle$:

(≤ 0) $[a, b] = 1$ commute, and hence $X = ab$.

(1) $[a, b]$ belongs to $\mathcal{E}$ and coincides with $[a_0, b]$ and with $[a, b_0]$ for every nontrivial $a_0 \in a$ and $b_0 \in b$; this subgroup is nontrivial and is contained in $Z(X)$, the center of X.

(2) For each nontrivial $a_0 \in a$ there exists a nontrivial $b_0 \in b$ such that $a^{b_0} = b^{a_0}$; and similarly with a and b interchanged.

Case (2) above is described as 'X is a rank one group with unipotent subgroups a and b.' Chapter I of [7] is concerned with the structure of such groups. The subgroups a and b as in (II)(2) are X-conjugate and their X-conjugacy class is called a *hyperbolic line*. Typical examples of X are the groups (P)SL$(2, k)$ for a (skewfield) k, in which case the hyperbolic line corresponds to the points of the projective line on which X acts 2-transitively.

Case (1) is the so-called special case; typical examples are extra-special p-groups of order p^3 and often suggestively denoted by p^{1+2}.

Case (≤ 0), the case where a and b commute, is indexed by the suggestive inequality because it will be partitioned into the following three subcases.

(-2) $a = b$, and so $X = a = b$.

(-1) $a \neq b$ and $X \setminus \{1\}$ is partitioned by $c \setminus \{1\}$ for $c \in \mathcal{E}$ with $c \leq X$. Here, we call the *line ab* the set of elements $c \in \mathcal{E}$ with $c \leq ab$. By $\mathcal{F}$ we denote the set of lines.

(0) $X \setminus \{1\}$ is not partitioned by $c \setminus \{1\}$ for $c \in \mathcal{E}$ with $c \leq X$.

For each i we shall write $\mathcal{E}_i$ to denote the relation on $\mathcal{E}$ expressing that a, b are in Case (i). So $(a, b) \in \mathcal{E}_{-2}$ is equivalent to $a = b$, and $a \in \mathcal{E}_{-1}(b)$ means that a and b belong to a line in $\mathcal{F}$. Notice that $\mathcal{E}_{\leq 0}(x)$ is the set of subgroups in $\mathcal{E}$ commuting with x.

In Definition (1.1) of [7], non-degeneracy of a set $\mathcal{E}$ of abstract root subgroups is defined as $\mathcal{E}_{\leq 0}$, $\mathcal{E}_1$, $\mathcal{E}_2$ being non-empty. The goal of this section is to prove that if $\mathcal{E}$ is a non-degenerate set of abstract root subgroups, then, under some reasonable restrictions, $(\mathcal{E}, \mathcal{F})$ is a non-degenerate root filtration space, see Theorem 13. Observe that, for $(a, b) \in \mathcal{E}_1$, we have $[a, b] \in \mathcal{E}$, so we have a map $[\cdot, \cdot] : \mathcal{E}_1 \to \mathcal{E}$ as required in the definition of root filtration space.

A priori it is not even clear that $(\mathcal{E},\mathcal{F})$ is a partial linear space. If $a,b\in\mathcal{E}$ are distinct points of the line cd, then $ab\le cd$ as groups, and so, by the partition property, also $ab\subseteq cd$ as lines in $\mathcal{F}$. But the reverse implication needs a proof.

We recall the following facts from [7].

Lemma 11 (Timmesfeld) *For a group G generated by a set $\mathcal{E}$ of abstract root subgroups, the following statements hold.*

- *(i) [Corollary II (2.3)] There are no triples $a,b,c\in\mathcal{E}$ with $a\in\mathcal{E}_{\le 0}(c)$ and $b\in\mathcal{E}_2(a)\cap\mathcal{E}_{-1}(c)$.*
- *(ii) [Lemma II (2.11)] Let $a,b,c\in\mathcal{E}$ with $ac\in\mathcal{F}$ and $b\in\mathcal{E}_2(a)\cap\mathcal{E}_i(c)$ for some $i\in\{1,2\}$. Then there is exactly one member of $\mathcal{E}_1(b)$ incident with ac and, for each $d\in ac$ (the line in $\mathcal{F}$) distinct from a and c, we have $ac=ad=cd$.*
- *(iii) [Lemma II (2.8)] Let $(a,b)\in\mathcal{E}_1$ and set $c=[a,b]$. If a is not isolated in $(\mathcal{E},\mathcal{E}_2)$, then $ac=\{c\}\cup\{a^{b_0}\mid b_0\in b\}\in\mathcal{F}$. Moreover, for each $a_0\in a\setminus\{1\}$, the map $b\to c$ given by $b_0\mapsto[a_0,b_0]$ is a bijection. In particular, if a and b are non-isolated points of $(\mathcal{E},\mathcal{E}_2)$, then $c\in\mathcal{E}_{-1}(a,b)$.*
- *(iv) [Exercise II (2.24)(1)(a)] Suppose that $\mathcal{E}$ is non-degenerate and that $(\mathcal{E},\mathcal{E}_2)$ has no isolated vertices. Then, for all $(a,b)\in\mathcal{E}_{-1}$, both $\mathcal{E}_2(a,b)$ and $\mathcal{E}_2(a)\cap\mathcal{E}_1(b)$ are nonempty.*

By $R(G)$ for a group G, we denote the solvable radical of G, that is, the biggest solvable normal subgroup of G. In Section II.4, Timmesfeld shows that this radical is nilpotent of class at most two when G is generated by a set of abstract root groups. However, we shall not use this in view of our restriction that $R(G)$ be trivial. Rather we need the following results in the case where $\mathcal{F}\neq\emptyset$.

Lemma 12 ([7] Theorem III (2.6)) *Let G be a group generated by a non-degenerate set $\mathcal{E}$ of abstract root subgroups. Assume that $(\mathcal{E},\mathcal{E}_2)$ is connected and that $R(G)=1$. Then for each $a\in\mathcal{E}$ the following assertions hold for $M_a:=\langle\mathcal{E}_{\le -1}(a)\rangle$.*

- *(i) $[M_a,M_a]=a$ for $a\in\mathcal{E}$.*
- *(ii) The set of isolated vertices of $(\mathcal{E}_{\le 0}(a),\mathcal{E}_2)$ coincides with $\mathcal{E}_{\le -1}(a)$.*
- *(iii) The point a is the single member of $\mathcal{E}$ commuting with all of M_a.*

Theorem 13 *Let G be a group. Suppose that $\mathcal{E}$ is a non-degenerate set of abstract root subgroups of G such that $(\mathcal{E},\mathcal{E}_2)$ is connected. If $R(G)=1$, then $(\mathcal{E},\mathcal{F})$, for $\mathcal{F}$ the set of lines of $\mathcal{E}$, is a non-degenerate root filtration space with thick lines.*

Proof. We first verify that $(\mathcal{E},\mathcal{F})$ is a partial linear space with thick lines. Since the direct product of two groups a, b contains more elements than $a\cup b$, lines are thick. Suppose that a and b are distinct collinear points of $\mathcal{E}$ and that c is a point incident with ab. By Lemma 11(iv) there is a point $d \in \mathcal{E}_2(a) \cap \mathcal{E}_1(b)$. By Lemma 11(i), all other points incident with ab are in $\mathcal{E}_2(d)$, and by (ii) of the same lemma, $ac = ab$. Repeated application of the observation gives that lines are uniquely determined by any two points they contain, so $(\mathcal{E}, \mathcal{F})$ is a partial linear space.

We next verify the conditions (A)–(H) of a non-degenerate root filtration space. Conditions (A) and (B) are trivially satisfied.

(E). Let $x \in \mathcal{E}$. We need to show that $\mathcal{E}_{\leq i}(x)$ is a subspace of $\mathcal{E}$ for $i = -1, 0$.

$i = -1$. This is the gamma space property. Let a, b, c, d be distinct points of $\mathcal{E}$ such that a, b, c are mutually collinear and d is incident with ab. Suppose that d is not collinear with c. Then, as d is a subgroup of ab which is contained in the centralizer in G of c, we have $d \in \mathcal{E}_0(c)$. By Lemma 12(ii), there exists $e \in \mathcal{E}_{\leq 0}(c) \cap \mathcal{E}_2(d)$. By Lemma 11(ii), there is a unique point in $\mathcal{E}_1(e)$ incident with ab such that all other points of ab are in $\mathcal{E}_2(e)$. But then either a or b lies in $E_2(e)$, which contradicts Lemma 11(i) (applied to the triple e, a, c or e, b, c).

$i = 0$. If $a, b \in \mathcal{E}_{\leq 0}(x)$, then a and b commute with x, whence so does every subgroup $c \in ab$, proving $c \in \mathcal{E}_0(x)$.

(C). Let u, v, x be points of $\mathcal{E}$ with $(u, v) \in \mathcal{E}_1$, and let i, j be such that $x \in \mathcal{E}_i(u) \cap \mathcal{E}_j(v)$ and $i \leq j$ (these restrictions on i and j do not harm the generality). By (II)(1), $[u, v]$ belongs to $\mathcal{E}$. We need to show that it is in $\mathcal{E}_{\leq i+j}(x)$.

First, suppose $i = -2$. Then, $u = x$, so $j = 1$. Then $[u, v] \in \mathcal{E}_{\leq -1}(u)$ follows from Lemma 11(iii).

Next, suppose $i = -1$. If $j = -1$. Then, by Lemma 12(i), as u, v both belong to M_x, we have $[u, v] \leq [M_x, M_x] = x$, proving $[u, v] = x \in \mathcal{E}_{-2}(x)$.

If $j = 0$, then by Lemma 11(iii) there is $v_0 \in v$ such that u^{v_0} is a point of the line joining u and $[u, v]$ distinct from u. Since $x^{v_0} = x$, it follows that x is collinear with both u and u^{v_0}, and so with each point of the line spanned by these two points. By the gamma space property, which has been shown to hold under (E) above, x is also collinear with $[u, v]$, so $[u, v] \in \mathcal{E}_{\leq -1}(x)$.

Suppose $j \geq 1$. In view of Lemma 11(i) applied to $[u, v]$, x, u, we have $[u, v] \in \mathcal{E}_{\leq 1}(x)$. If $j = 2$, there is nothing left to prove. Therefore, we may assume, without loss of generality, that $j = 1$. Suppose $[u, v] \in \mathcal{E}_1(x)$. Then, by Lemma 11(iii), $u = [[u, v], x]$. Now take $x_0 \in x \setminus \{1\}$. By Lemma 11(iii), $[u, v]^{x_0}$ and v^{x_0} are collinear points of the line joining u and $[u, v]$ and the line joining v and

$[v, x]$, respectively. Moreover, $[u, v]^{x_0}$ is distinct from $[u, v]$ and u, while v^{x_0} is distinct from v and $[v, x]$. Now v^{x_0} is collinear with both $[u, v]^{x_0}$ and v, which is possible only if $v^{x_0} = [u, v]$, and hence $[u, v] = [x, v]$ as well. Consequently, $[u, v] \in \mathcal{E}_{\leq 0}(x)$.

Next we consider the case where $i = 0$. If $j \geq 2$, there is nothing to show, so we may assume $j \in \{0, 1\}$. If $j = 0$, then both u and v commute with x, and so also $[u, v]$ commutes with x, proving $[u, v] \in \mathcal{E}_{\leq 0}(x)$. Thus, it remains to study $j = 1$. But then Lemma 11(i), applied to x, $[u, v]$, u, shows that $[u, v] \in \mathcal{E}_{\leq 1}(x)$.

(D). This is immediate from Lemma 11(i).

(F). Fix $x \in \mathcal{E}$. We need to show that $\mathcal{E}_{\leq 1}(x)$ is a proper hyperplane of $\mathcal{E}$. Let $(a, b) \in \mathcal{E}_{-1}$. By Lemma 11(ii), $\mathcal{E}_{\leq 1}(x)$ contains at least one point of ab. Since connectedness of $(\mathcal{E}, \mathcal{E}_2)$ implies that $\mathcal{E}_{\leq 1}(x)$ of is a proper subset of $\mathcal{E}$, it remains to show that $\mathcal{E}_{\leq 1}(x)$ is a subspace. To this end, assume $a, b \in \mathcal{E}_{\leq 1}(x)$ (and still $ab \in \mathcal{F}$). Since $y \in ab$ with $y \in \mathcal{E}_2(x)$, would contradict Lemma 11(ii), we find $ab \subseteq \mathcal{E}_{\leq 1}(x)$.

(G). As $(\mathcal{E}, \mathcal{E}_2)$ is connected, $\mathcal{E}$ has no isolated vertices unless $|\mathcal{E}| = 1$. But in that case G is abelian and hence $R(G) = G$.

(H). By the hypothesis on $(\mathcal{E}, \mathcal{E}_2)$ and Lemma 5(ii), the condition (H) is satisfied. This ends the proof of the theorem. □

The properties used from [7] are those of Lemmas 11 and 12. These are proven at an early stage of the treatment of non-degenerate sets of abstract root subgroups in [7]. In this sense, Theorem 9 supplies an alternative proof of Timmesfeld's classification.

The converse of Theorem 13 does not hold. To see this, consider a thick building of an irreducible spherical type having multiple bonds. Then there are two root shadow spaces of this building, according to two distinct interpretations of the Coxeter diagram as a Dynkin diagram. Except for 'bad characteristics', only one of these choices will lead to a correspondence with long root subgroups, and hence lead to abstract root subgroups.

4 Lie algebras

In this section, we study Lie algebras generated by a particular kind of elements, called extremal. Under certain non-degeneracy conditions, we are able to find the structure of a root filtration space on the projective points corresponding to these elements. We also exhibit a relation with abstract root subgroups.

Let L be a Lie algebra over the field k. In [4], extremal elements are defined if the characteristic of k is not 2, and useful identities, due to Premet, are derived. In order to extend our results to characteristic 2, we incorporate Premet's identities as (2) and (3) into the definition.

Definition 14 *An element $x \in L$ is* extremal *if there is a map $g_x : L \to k$ such that*

$$[x,[x,y]] = 2g_x(y)x, \tag{1}$$

$$[[x,y],[x,z]] = g_x([y,z])x + g_x(z)[x,y] - g_x(y)[x,z], \tag{2}$$

and

$$[x,[y,[x,z]]] = g_x([y,z])x - g_x(z)[x,y] - g_x(y)[x,z] \tag{3}$$

hold for every $y, z \in L$.

By the Jacobi identity,

$$[x,[y,[x,z]]] - [[x,y],[x,z]] = [y,[x,[x,z]]],$$

so the combination of (2) *and* (1) is equivalent to the combination of (1) *and* (3). If $\mathrm{char}\, k \neq 2$, then (2) and (3) follow from (1), so the current definition extends the one of [4]. We shall write E for the set of nonzero extremal elements of L and $\mathcal{E}$ for the corresponding set of projective points, so $\mathcal{E} = \{kx \mid x \in E\}$. Recall from [9] that an element $x \in L$ is a *sandwich* in L if $\mathrm{ad}_x^2 = 0$ and $\mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x = 0$ for every $y \in L$. Thus a sandwich is an extremal element x for which g_x can be chosen to be identically zero. If $\mathrm{char}\, k \neq 2$ then every element x with $\mathrm{ad}_x^2 = 0$ is automatically a sandwich.

The usefulness of properties (1) and (2) mainly relies on the following fact. For $x \in L$ and $t \in k$ define the map $\exp(x,t) : L \to L$ by

$$\exp(x,t)y = y + t[x,y] + t^2 g_x(y)x.$$

Lemma 15 *Let $x \in L$ be such that $\mathrm{ad}_x^3 = 0$ and let $g_x : L \to k$ be a function. Then the map $\exp(x,t)$ is an endomorphism of L for every $t \in k$ if g_x is k-linear, and (2) and $g_x(z)\mathrm{ad}_x^2 y = g_x(y)\mathrm{ad}_x^2 z$ hold for every $y, z \in L$. Furthermore, $\exp(x,s)\exp(x,t) = \exp(x,s+t)$ for every $s, t \in k$ if (1) and $g_x(x) = g_x([x,y]) = 0$ hold for every $y \in L$.*

If the field k has more than two elements, the converse of each of the latter two assertions also holds.

Proof. Obviously, $\exp(x,t)$ is a linear map if and only if g_x is. If so, $\exp(x,t)$ is an endomorphism if and only if $\exp(x,t)[y,z] = [\exp(x,t)y, \exp(x,t)z]$ for every $y,z \in L$, and hence the first statement follows from the formula

$$\begin{aligned}&[\exp(x,t)y, \exp(x,t)z] - \exp(x,t)[y,z] = \\ &\quad t^2(g_x(y)[x,z] + g_x(z)[y,x] + [[x,y],[x,z]] - g_x([y,z])x) + \\ &\quad t^3(g_x(z)[[x,y],x] + g_x(y)[x,[x,z]]).\end{aligned}$$

Likewise, the second assertion follows from

$$\begin{aligned}&\exp(x,s)\exp(x,t)y - \exp(x,s+t)y = \\ &\quad st([x,[x,y]] - 2g_x(y)x) + s^2tg_x([x,y])x + s^2t^2g_x(y)g_x(x)x.\end{aligned}$$

The last statement follows from the above formulae and the fact that the polynomial functions $t \mapsto t^2$ and $t \mapsto t^3$ are distinct for $|k| > 2$. □

In characteristic 2, formulas (1) and (2) do not uniquely define the function g_x. However, we have the following.

Lemma 16 *Let $x \in L$. Then $N_L(kx)$ has codimension at most one in L (or, equivalently, $[x,L] \subseteq kx + k[x,y]$ for some $y \in L$) if any of the following holds.*

- *(i)* $\mathrm{ad}_x^2 L = kx$ *and* $\mathrm{char}\, k = 2$.
- *(ii) There are two distinct functions g_x and g'_x with property (2).*
- *(iii) There is a function g_x satisfying (2), but g_x is not k-linear or g_x is not identically zero on $C_L(x) + [x,L]$. In these cases kx is an ideal.*

Furthermore, if $\mathrm{char}\, k = 2$, *and $x \in E$ is such that $N_L(kx)$ has codimension at most one in L, then x is a sandwich.*

Proof. Assume that (i) holds. Then $[x,[x,y]] = f_x(y)x$ for some nonzero linear function $f_x : L \to L$. By [4], Lemma 2.2,

$$0 = 2[[x,y],[x,z]] = f_x([y,z])x + f_x(z)[x,y] - f_x(y)[x,z].$$

for every $y,z \in L$. Choose $y \in L$ with $f_x(y) \neq 0$. Then

$$[x,z] = f_x(y)^{-1}(f_x([y,z])x + f_x(z)[x,y])$$

for every $z \in L$. Therefore, $[x,L] \subseteq kx + k[x,y]$.

If (ii) holds then, putting $h_x = g_x - g'_x$, we find

$$0 = h_x([y,z])x + h_x(z)[x,y] - h_x(y)[x,z].$$

Again, with an arbitrary $y \in L$ such that $h_x(y) \neq 0$ we have $[x,L] \subseteq kx + k[x,y]$.

Concerning case (iii), assume that g_x satisfies (2). Suppose that g_x is not additive. Then there are y_1, y_2 in L such that $g_x(y_1+y_2) \neq g_x(y_1) + g_x(y_2)$. The following calculation for arbitrary $z \in L$ shows that $[x,L] \subseteq kx$.

$$\begin{aligned}
0 &= [[x,y_1+y_2],[x,z]] - [[x,y_1][x,z]] - [[x,y_2],[x,z]] \\
&= (g_x([y_1+y_2,z]) - g_x([y_1,z]) - g_x([y_2,z])x \\
&\quad +(g_x(y_1+y_2) - g_x(y_1) - g_x(y_2))[x,z]
\end{aligned}$$

The proof for the remaining cases of (iii) is similar.

To see the last assertion, assume that $x \in E$ and $y \in L$ satisfy $[x,L] \subseteq kx + k[x,y]$ and that $\operatorname{char} k = 2$. Then $\mathrm{ad}_x^2 = 0$ by (1). Moreover, as $[x,[x,y]] = 0$, the subspace $kx + k[x,y]$ is a commutative Lie subalgebra, and so $[x,L]$, being a subspace of $kx + k[x,y]$, is commutative also. Therefore, (2) and (3) imply $[x,[L,[x,L]]] = [[x,L],[x,L]] = 0$, whence x is a sandwich. □

Convention. We insist that g_x is identically zero whenever x is a sandwich.

By Lemma 16, for each $x \in E$, this convention turns g_x into a uniquely defined function on L, which is linear.

Lemma 17 *Let $x \in E$ and $y,z \in L$. Then*

$$((\mathrm{ad}_x\mathrm{ad}_y)^2 + g_x(y)(\mathrm{ad}_x\mathrm{ad}_y))z = g_x([y,[y,z]])x - g_x([y,z])[x,y], \tag{4}$$

and

$$((\mathrm{ad}_y\mathrm{ad}_x)^2 + g_x(y)(\mathrm{ad}_y\mathrm{ad}_x))z = g_x(z)[y,[y,x]] - g_x([y,z])[x,y]. \tag{5}$$

Proof. Applying Formula (3) with $[y,z]$ instead of z gives (4). The other statement also follows from (3): apply ad_y to both sides. □

Lemma 18 *Assume that $x \in E$ and $y \in L$ satisfy $[x,y] = 0$. Then $g_x(y) = 0$ and $g_x([y,z]) = 0$ for every $z \in L$.*

Proof. By (2), $g_x(y)[x,z] = g_x([y,z])x$. Clearly, there is nothing to show if $g_x(y) = 0$. If $g_x(y) \neq 0$, this shows that x is a sandwich, so $g_x(y) = 0$ by convention, a contradiction. □

Lemma 19 *Let $x, y \in E$ and $z \in L$. Then*

$$g_x(y) = g_y(x), \tag{6}$$

and

$$g_x([y,z]) = -g_y([x,z]). \tag{7}$$

Proof. Lemma 18 asserts both statements in the case where $[x,y] = 0$. Assume that x and y do not commute. Interchange x and y in (5), and subtract the equality from (4). By (1), we obtain

$$(g_x(y) - g_y(x))[x,[y,z]] = -(g_x([y,z]) + g_y([x,z]))[x,y].$$

This gives both results in the case where $[x,[y,z]]$ and $[x,y]$ are linearly independent. Furthermore, if we prove in the general case that $g_x(y) = g_y(x)$ then the second equality automatically follows (because $[x,y] \neq 0$). Based on (1), this is easy if $\operatorname{char} k \neq 2$. We also succeed if there exists an element z' such that $[x,[y,z']]$ and $[x,y]$ are linearly independent, or, by interchanging the roles of x and y, if $[y,[x,z']]$ and $[x,y]$ are linearly independent. Thus it remains to give a proof for $\operatorname{char} k = 2$ and $[x,[y,L]] + [y,[x,L]] \subseteq k[x,y]$. Since $[y,[x,y]] = 0$, it follows that $[y,[x,[y,L]]] = 0$. But (3) shows that in this case $[y,L] \subseteq ky + k[x,y]$, so by Lemma 16, y is a sandwich and hence, by convention, g_y is identically zero. Similarly, $[x,L] \subseteq kx + k[x,y]$ and hence g_x is identically zero as well. □

Proposition 20 *Suppose that L is generated by E. Then L is linearly spanned by E and there is a unique bilinear form $g : L \times L \to k$ such that for every $x \in E$ and for every $y \in L$, $g(x,y) = g_x(y)$. The form g is symmetric and associative.*

Proof. Linear spanning follows as in [4] (an argument based on the observation that, by Lemma 15, $\exp(x,1)$ is an automorphism, so that $\exp(x,1)y = y + [x,y] + g_x(y)x$ is again extremal and so $[x,y] = \exp(x,1)y - y - g_x(y)x$ lies in the linear span of E). Existence of the form g can be proved using linearity of g_x and formula (6). The symmetry and associativity follow from (6) and (7). □

Lemma 21 *For $x \in E$ and $y \in L$ such that $g_x(y) \neq 0$, the following assertions hold.*

(i) *x, y and $[x,y]$ are linearly independent. In particular, if $y \in E$ then the Lie subalgebra generated by x and y is isomorphic to* $\mathrm{sl}(2,k)$.

(ii) *For every $z \in N_L(x)$ we have $g_x(z) = 0$ and $[z,x] = g_x([y,z])g_x(y)^{-1}x$.*

(iii) *If there is an element $z \in L$ with $g_x([y,z]) \neq 0$, then $x \in [x,L]$.*

Proof. (i). First we show that x and $[x,y]$ are linearly independent. Assume that $[x,y] = \lambda x$ for some $\lambda \in k$. Then, from (1) and (2) we derive, for each $z \in L$,

$$2\lambda g_x(z)x = \lambda[x,[x,z]] = [[x,y],[x,z]] = g_x([y,z])x + g_x(z)[x,y] - g_x(y)[x,z].$$

Since $g_x(y) \neq 0$, this implies that $[x,L] \subseteq k\,x + k[x,y]$, so by Lemma 16 x is a sandwich and by convention g_x is identically zero, a contradiction. Furthermore, $y \notin kx + k[x,y]$ because by Lemma 18, $g_x(x) = g_x([x,y]) = 0$ and g_x is linear. So $g_x(y)^{-1}x$, $g_x(y)^{-1}[x,y]$, and y are linearly independent. If $y \in E$ then they are easily verified to be a Chevalley basis of $\mathrm{sl}(2,k)$.

(ii). Assume that $[z,x] = \alpha x$ with $\alpha \in k$. Then formulas (1) and (2) give

$$2\alpha g_x(y)x = [[x,y],[x,z]] = g_x([y,z])x + g_x(z)[x,y] + \alpha g_x(y)x,$$

whence $(g_x([y,z]) - \alpha g_x(y))x + g_x(z)[x,y] = 0$. By (i), x and $[x,y]$ are linearly independent, so $g_x(z) = 0$ and $g_x([y,z]) = \alpha g_x(y)$.

(iii). For $z \in L$, by (3),

$$g_x([y,z])x = [x,[y,[x,z]]] + g_x(y)[x,z] + g_x(z)[x,y] \in [x,L].$$

□

Proposition 22 *Suppose that $x,y \in E$ satisfy $g_x(y) = 1$. Then there is a $\mathbb{Z}$-grading*

$$L = L_{-2}(x,y) + L_{-1}(x,y) + L_0(x,y) + L_1(x,y) + L_2(x,y), \tag{8}$$

with $L_{-2}(x,y) = kx$, $L_2(x,y) = ky$, $L_0(x,y) = N_L(kx) \cap N_L(ky)$, $L_{-1}(x,y) = [x,U]$, and $L_1(x,y) = [y,U]$, where

$$U = \{u \in L \mid g_x(u) = g_y(u) = g_x([y,u]) = 0\}.$$

Furthermore, ad_x induces a linear isomorphism $L_1(x,y) \to L_{-1}(x,y)$ with inverse $-\mathrm{ad}_y$. For each $i \in \{-2,-1,0,1,2\}$, the subspace $L_i(x,y)$ is contained in the i-eigenspace of $\mathrm{ad}_{[x,y]}$.

Proof. Set $S = kx + ky + k[x,y]$, so $S \cong \mathrm{sl}(2,k)$ by Lemma 21(i). Put $h = [x,y]$.

We first show the last assertion. Since $[h,x] = -2g_x(y)x = -2x$, the statement holds for $i = -2$, and similarly by (6), we have $[h,y] = 2g_y(x)y = 2y$, whence the statement for $i = 2$. If $v \in L_{-1}(x,y)$, there is $u \in U$ with $v = [x,u]$. Now by (3), as $g_x([y,u]) = g_x(u) = 0$,

$$[h,v] = [[x,y],[x,u]] = -[x,u] = -v,$$

so indeed v is a -1-eigenvector of ad_h. The argument for $i = 1$ is similar. If $v \in L_0(x,y)$, then by Lemma 21(ii) $[v,x] = -g_x([y,v])x$ and $[v,y] = -g_y([x,v])y$, so by Jacobi and (7) $[h,v] = [[x,y],v] = [[v,x],y] + [x,[v,y]] = -g_x([y,v])[x,y] - g_y([x,v])[x,y] = 0$, proving that v lies in the kernel of ad_h. This establishes the last assertion of the proposition.

Let $u \in U$. By Lemma 18, $g_x([x,u]) = 0$. Formula (7) gives $g_y([x,u]) = -g_x([y,u]) = 0$ and $g_x([y,[x,u]]) = -g_y([x,[x,u]]) = -2g_x(u)g_y(x) = 0$. Thus U is ad_x-invariant. A similar argument shows that U is ad_y-invariant as well, so U is S-invariant.

Next we determine $U \cap S$. If $u \in U \cap S$, then there are α, β, γ in k with $u = \alpha x + \beta y + \gamma h$ and $g_x(u) = g_y(u) = g_y([x,u]) = 0$. By Lemma 18, $g_x(x) = g_x(h) = 0$ and $g_y(y) = g_y(h) = 0$, so the three equations give $\alpha = \beta = 0$, proving $u \in k\,h$. But $g_y([x,h]) = 2$, so $S \cap U = 0$ unless $\mathrm{char}\, k = 2$ in which case $S \cap U = k\,h$.

By (3), $\mathrm{ad}_x\mathrm{ad}_y\mathrm{ad}_x z = -\mathrm{ad}_x z$ and $\mathrm{ad}_y\mathrm{ad}_x\mathrm{ad}_y z = -\mathrm{ad}_y z$ for every $z \in U$. It follows that ad_x and ad_y are linear isomorphisms between $L_1(x,y)$ and $L_{-1}(x,y)$ with inverses $-\mathrm{ad}_y$ and $-\mathrm{ad}_x$, respectively. By (1), $\mathrm{ad}_x^2 U = \mathrm{ad}_y^2 U = 0$. This, together with the observation above, implies $L_1(x,y) \cap L_{-1}(x,y) = 0$.

We also see that $\mathrm{ad}_x U = \mathrm{ad}_x\mathrm{ad}_y U$ and $\mathrm{ad}_y U = \mathrm{ad}_y\mathrm{ad}_x U$. Lemma 18 implies $C_L(S) \subseteq U$. Hence $C_L(S) = C_U(S)$. For $z \in U$ consider the element $u(z) = z + \mathrm{ad}_x\mathrm{ad}_y z + \mathrm{ad}_y\mathrm{ad}_x z$. We have $\mathrm{ad}_x u(z) = \mathrm{ad}_x z + \mathrm{ad}_x^2\mathrm{ad}_y z + \mathrm{ad}_x\mathrm{ad}_y\mathrm{ad}_x z = \mathrm{ad}_x z - \mathrm{ad}_x z = 0$ and similarly $\mathrm{ad}_y u(z) = 0$. Thus $u(z) \in C_U(S)$ and the decomposition $z = u(z) - \mathrm{ad}_x\mathrm{ad}_y z - \mathrm{ad}_y\mathrm{ad}_x z$ gives the direct decomposition $U = C_L(S) + L_{-1}(x,y) + L_1(x,y)$. It is also obvious that $L_{-1}(x,y)$ and $L_1(x,y)$ are invariant under the adjoint action of $C_L(S)$.

Let M be the sum of all $L_i(x,y)$ for $i = -2,\ldots,2$. We claim that $L = M$. Clearly, $S \subseteq M$ and $C_L(S) \subseteq L_0(x,y)$, so, by the above decomposition of U, also $U \subseteq M$. By the definition of U, the codimension of U in L is at most 3. Since $U \cap (kx + ky) = 0$, the codimension is at least 2. If the codimension is 2 or if the codimension is 3 and $S \cap U = 0$, then $L = U + S$ and the claim follows. Otherwise, there exists $z \in L$ with $g_x([y,z]) \neq 0$, such that $L = U + S + kz$. We will establish that z can be chosen to lie in $L_0(x,y)$. By (7), we have $g_y([x,z]) = -g_x([y,z]) \neq 0$. Hence, by Lemma 21(iii) there are

$v, w \in L$ with $x = [x, v]$ and $y = [y, w]$. In view of Lemma 21(ii), $g_x([y, w]) = -1 = g_y([x, v])$. This shows that both v and w span complements to $U + S$ in L. The decomposition $L = U + S + kw = C_L(S) + L_{-1}(x, y) + L_1(x, y) + S + kw$ gives

$$[y, L] = [y, L_{-1}(x, y) + kx + kw].$$

In particular, $[y, v] = [y, \alpha x + \beta u + \gamma w]$ for certain $\alpha, \beta, \gamma \in k$ and $u \in L_{-1}(x, y)$. By Lemma 18 and the above, $-1 = g_x([y, v]) = \gamma g_x([y, w]) = -\gamma$, so $\gamma = 1$. Now $b = v - \alpha x - \beta u$ satisfies $[y, b] = [y, w] = y$ and $[x, b] = [x, v] = x$, so $b \in L_0(x, y)$. Since b also spans a complement to $U + S$ in L, we find $L = M$, as claimed.

The action of ad_b on S coincides with the action of $(1/2)\mathrm{ad}_h$ for $\mathrm{char}\, k \neq 2$, so that, in all characteristics, either $L_0(x, y) = C_L(S)$ or there exists $z \in L$ with $[x, z] = x$ and $[y, z] = -y$ such that $L_0(x, y) = C_L(S) + kz$.

The next step is to show that L is the direct sum of the subspaces $L_i(x, y)$ where $i \in \{-2, -1, 0, 1, 2\}$. As each $L_i(x, y)$ is contained in the i-eigenspace of ad_h, this is immediate for $\mathrm{char}\, k > 3$. If $\mathrm{char}\, k = 3$, we need to verify that $L_{\epsilon 1}(x, y) \cap L_{\epsilon 2}(x, y) = 0$ for $\epsilon = \pm$. Since, as we have seen above, neither x nor y belongs to $S \cap U$, and $L_{\epsilon 1}(x, y)$ is contained in U, we must have $L_{\epsilon 1}(x, y) \cap L_{\epsilon 2}(x, y) = 0$ in all characteristics. Hence we may assume $\mathrm{char}\, k = 2$ and need to verify $L_1(x, y) \cap L_{-1}(x, y) = 0$. But this has been shown above.

Finally, we prove $[L_i(x, y), L_j(x, y)] \subseteq L_{i+j}(x, y)$ for all i, j, that is, the $L_i(x, y)$ are a $\mathbb{Z}$-grading of L. By the same argument as above, this is immediate if $\mathrm{char}\, k > 3$. Nevertheless, we prove the statements for all characteristics. Without loss of generality, we may take $i \leq j$. Also, by symmetry of the roles of x and y, we may take (i, j) to be lexicographically smaller than or equal to $(-j, -i)$, whence $i + j \leq 0$. For $i = -2$, the inclusion follows from the definition of U and extremality of x.

Consider the case $i = -1$. If $j = -1$, the inclusion follows from (2).

Let $j = 0$. If $z \in C_L(S)$ then, by Lemma 18, $g_x([z, u]) = g_y([z, u]) = 0$ and $g_x([y, [z, u]]) = g_x([z, [y, u]]) = 0$ for every $u \in L$ and hence $[z, U] \subseteq U$. If $[x, z] = x$ and $[y, z] = -y$ then by (2) for every $u \in U$ we have $0 = [x, [x, u]] = [[x, z], [x, u]] = g_x([z, u])x + g_x(u)[x, z] - g_x(z)[x, u] = g_x([z, u])x$ whence $g_x([z, u]) = 0$. As U is ad_y-invariant, we also have $g_x([y, [z, u]]) = g_x([z, [y, u]]) + g_x([[y, z], u]) = 0 - g_x([y, u]) = 0$ for every $u \in U$. By symmetry, $g_y([z, u]) = 0$ also holds for every $u \in U$, whence $\mathrm{ad}_z U = U$. As $L_0(x, y) = C_L(S) + kz$, we conclude $[L_0(x, y), U] \subseteq U$. Now $L_{-1}(x, y) = [x, U]$, so, by Jacobi,

$$\begin{aligned}&[L_0(x, y), L_{-1}(x, y)] = [L_0(x, y), [x, U]]\\ &\subseteq [[L_0(x, y), x], U] + [x, [L_0(x, y), U]] \subseteq [x, U] = L_{-1}(x, y).\end{aligned}$$

This settles $j = 0$.

Suppose $j = 1$. Every element of $L_{-1}(x, y)$ is of the form $[x, u]$ for some $u \in U$, while every element of $L_1(x, y)$ is of the form $[y, [x, v]]$ with $v \in U$. Now

$$\begin{aligned}
[x, [[x, u], [y, [x, v]]]] &= [[x, [x, u]], [y, [x, v]]] + [[x, u], [x, [y, [x, v]]]] \\
&= 2g_x(u)[x, [y, [x, v]]]] + \\
&\quad [[x, u], g_x([y, v])x - g_x(v)[x, y] - g_x(y)[x, v]] \\
&= -[[x, u], [x, v]] = -g_x([u, v])x.
\end{aligned}$$

This shows $[L_{-1}(x, y), L_1(x, y)] \subseteq L_0(x, y)$, settling the case $j = 1$.

It remains to deal with $(i, j) = (0, 0)$, but this is immediate from the fact that $L_0(x, y)$ is a Lie subalgebra of L. □

Corollary 23 *Assume that $x, y \in E$ satisfy $g_x(y) \neq 0$. Then there is a filtration*

$$kx = L_{\leq -2}(x) \subseteq L_{\leq -1}(x) \subset L_{\leq 0}(x) \subseteq L_{\leq 1}(x) \subset L_{\leq 2}(x) = L, \tag{9}$$

where $L_{\leq i}(x) = \sum_{j=-2}^{i} L_j(x, y)$. The subspaces $L_{\leq i}(x)$ are independent of the choice of y:

$$L_{\leq 1}(x) = \{z \in L \mid g_x(z) = 0\},\ L_{\leq 0}(x) = N_L(kx),\ L_{\leq -1}(x) = kx + [x, L_{\leq 1}(x)].$$

We shall study affine counterparts E_i of the relations $\mathcal{E}_i$ to be introduced later. These are relations on E, defined as follows.

(−2) E_{-2} is linear dependence between members of E.

(−1) E_{-1} is defined by $(x, y) \in E_{-1}$ if and only if x and y are linearly independent, $[x, y] = 0$ and, for every $z \in L$,

$$[x, [y, z]] = g_y(z)x + g_x(z)y. \tag{10}$$

(0) E_0 stands for commuting, but not in $E_{-2} \cup E_{-1}$.

(1) E_1 is defined by $(x, y) \in E_1$ if and only if $g_x(y) = 0$ and $[x, y] \neq 0$.

(2) E_2 consists of all pairs $(x, y) \in E \times E$ with $g_x(y) \neq 0$.

Lemma 24 *Let $(x, y) \in E_{-1}$. Then for every pair $(\lambda, \mu) \in k^2 \setminus \{(0, 0)\}$, we have $\lambda x + \mu y \in E$, with $g_{\lambda x + \mu y} = \lambda g_x + \mu g_y$ and $(x, \lambda x + \mu y) \in E_{-1}$.*

Proof. Since $\lambda x, \mu y \in E$ with $g_{\lambda x} = \lambda g_x$ and $g_{\mu y} = \mu g_y$, it suffices to prove the assertions for $\lambda = \mu = 1$.

First we show that $x+y \in E$ with $g_{x+y} = g_x + g_y$. Straightforward expansion, using $g_x(y) = g_x([y,z]) = 0$ (which is a consequence of Lemma 18) and (10), gives

$$\exp(x,t)\exp(y,t)z = z + t[x+y,z] + t^2(g_x(z) + g_y(z))(x+y),$$

and, similarly, $\mathrm{ad}_{x+y}^3 = 0$. Hence, the map $\exp(x+y,t) : z \mapsto z + t[x+y,z] + t^2(g_x(z) + g_y(z))(x+y)$ is an automorphism of L and Lemma 15 applies with $g_{x+y} = g_x + g_y$, Formula (10) implies that $\exp(x+y,t)$ is additive in t.

We need to show that this definition of g_{x+y} is consistent with our convention in the characteristic 2 case. Assume therefore that $\mathrm{char}\, k = 2$ and there exists a function $g'_{x+y} \neq g_{x+y}$ such that $x+y$ satisfy the definition of extremality with fuction g'_{x+y} as well. According to Lemma 16 in this case $x+y$ is a sandwich and hence we need to show that $g_{x+y} = 0$. Lemma 16(ii) gives that $[x+y,L] \subseteq k(x+y) + k[x+y,z]$ for some element $z \in L$. By (10), $[x+y,[y,L]] = [x,[y,L]] \subseteq kx + ky$. On the other hand, $[x+y,[y,L]] \subseteq [x+y,L] \subseteq k[x+y] + k[x+y,z]$. Assume first that $[x+y,z] \in kx+ky$. Then $[x,[y,L]] = [x,[x+y,L]] \subseteq [x, kx+ky] = (0)$ whence $g_x = g_y = 0$. Otherwise $k(x+y) + k[x+y,z] \cap kx + ky = k(x+y)$ whence $[x,[y,L]] \subseteq k(x+y)$ which implies $g_x + g_y = 0$. In either case we have $g_{x+y} = 0$, as required.

The property $(x, x+y) \in E_{-1}$ follows easily from linearity of Formula (10). □

Lemma 25 *For $(x,y) \in E_1$ the following assertions hold.*

(i) $[x,y] \in E$ *and* $g_{[x,y]}(z) = g_x([y,z])$ *for every* $z \in L$.
(ii) $(x,[x,y]) \in E_{-1}$.
(iii) $y \notin N_L(kx)$.

Proof. (iii). If $y \in N_L(x) \setminus C_L(x)$ then $[y,[y,x]] \in k^*x$, contradicting $g_y(x) = 0$.

We prove statements analogous to (i) and (ii) for $x + [x,y]$ in place of $[x,y]$. Then Lemma 24 gives (i) and (ii) for $[x,y]$.

(i). From Lemma 15 we see that $x + [x,y] = \exp(y,-1)x \in E$ with $g_{x+[x,y]}(z) = g_x(\exp(y,1)z) = g_x(z) + g_x([y,z]) + g_x(g_y(z)y) = g_x(z) + g_x([y,z])$.

(ii). Clearly, $[x, x+[x,y]] = 0$ as $g_x(y) = 0$. By Jacobi, (1), (2), and (i),

$$\begin{aligned}
[x,[x+[x,y],z]] &= [x,[x,z]] + [[x,[x,y]],z] + [[x,y],[x,z]] \\
&= 2g_x(z)x + g_x([y,z])x + g_x(z)[x,y] \\
&= g_{x+[x,y]}(z)x + g_x(z)(x + [x,y]).
\end{aligned}$$

We conclude that (10) holds with $x+[x,y]$ instead of y, whence (ii). $\square$

Corollary 26 *Assume that L is generated by E. Then the subalgebra generated by the sandwiches of L is an ideal of L.*

We have the following converse of Lemma 24.

Lemma 27 *Let $x, y \in E$ be linearly independent. Assume that L is generated by E and x is not a sandwich. If $\lambda x + \mu y \in E$ for some $\lambda, \mu \in k^*$, then $(x, y) \in E_{-1}$.*

Proof. From the existence of g as in Proposition 20 we infer that if $\lambda x+\mu y \in E$ then $g_{\lambda x+\mu y} = \lambda g_x + \mu g_y$. We claim that $\lambda x + \mu y \in E \cup \{0\}$ for $\lambda, \mu \in k$ if and only for every $u, z \in L$ both of the following equations hold.

$$\lambda\mu([x,[y,z]] + [y,x,z]) = 2\lambda\mu(g_y(z)x + g_x(z)y)$$

and

$$\begin{aligned}\lambda\mu([x,[u,[y,z]]] + [y,[u,[x,z]]]) = {}&\lambda\mu(g_x([u,z])y + g_y([u,z])x + g_x(z)[y,u]\\ &+g_y(z)[x,u] + g_x(u)[y,z] + g_y(u)[x,z]).\end{aligned}$$

Indeed, by expanding Formula (1) with $\lambda x+\mu y$ instead of x, replacing y by z, and subtracting the corresponding formulas for $\lambda x \in E\cup\{0\}$ and $\mu y \in E\cup\{0\}$, we find that the first equation is equivalent to condition (1) for $\lambda x + \mu y$. Likewise, by expanding Formula (3) for $\lambda x+\mu y$ instead of x, replacing y by u, and subtracting the corresponding formulas for $\lambda x \in E\cup\{0\}$ and $\mu y \in E\cup\{0\}$, we obtain that the second equation is equivalent to condition (3), whence the claim.

From the claim it is immediate that if $\lambda x+\mu y \in E$ for some $\lambda \neq 0, \mu \neq 0$ then the whole projective line spanned by x and y is in $\mathcal{E}$. By the linear spanning of L by E (see Proposition 20) and the assumption that x is not a sandwich, there exists $v \in E$ such that $g_x(v) = 1$. Let $a \in kx+ky\setminus\{0\}$ be such that $g_v(a) = 0$. Then $a \in E$ by what we have seen above. Now $[a, v] \in E$ and $(a, [a, v]) \in E_{-1}$ by Lemma 25(ii). By Corollary 23, $a = [x, [a, v]]$ and $g_x([a, v]) = 0$. Lemma 25(ii) gives then $(x, a) \in E_{-1}$, from which, by Lemma 24, we conclude that $(x, y) \in E_{-1}$. $\square$

Theorem 28 *Suppose that L is a Lie algebra over k generated by E. Then the following hold.*

(i) The set of subgroups $\{\exp(x,t) \mid t \in k\}$ of $\mathrm{Aut}(L)$ for $x \in E$ is a set of abstract root subgroups of the subgroup of $\mathrm{Aut}(L)$ which they generate.

(ii) Let $\mathcal{E}$ be the set of projective points spanned by extremal elements of L and $\mathcal{F}$ the set of projective lines all of whose points belong to $\mathcal{E}$. If L does not contain sandwiches then $(\mathcal{E}, \mathcal{F})$ is a root filtration space. Furthermore, let $\mathcal{B}_i$ $(i \in I)$ be the connected components of $(\mathcal{E}, \mathcal{E}_2)$. Then each $\mathcal{B}_i$ is either a non-degenerate root filtration space or a root filtration space with an empty set of lines. Furthermore, L is the direct sum of the Lie subalgebras generated by $\mathcal{B}_i$.

Proof. (i) can be proved as in [4].

(ii). We verify the axioms (A)–(F). Axiom (A) holds by definition of $\mathcal{E}_2$ and (B) by Lemma 27.

(C). Suppose $(u, v) \in E_1$. Then $[u, v] \in E$ by Lemma 25(i), and $[ku, kv] = k[u, v] \in \mathcal{E}$. Suppose now $x \in \mathcal{E}_i(u) \cap E_j(v)$. We need to show $k[u, v] \in \mathcal{E}_{\leq i+j}(kx)$. This will follow from the filtration in Corollary 23 once we prove the claim:

$$E_l(x) = (E \cap L_{\leq l}(x)) \setminus L_{\leq l-1}(x),$$

where $L_{\leq l}(x)$ is defined as in Corollary 23 for $l \in \{-2, -1, 0, 1, 2\}$ and empty otherwise.

To substantiate the claim, we only need verify $E_{\leq l}(x) = E \cap L_{\leq l}(x)$. For $l = -2, 1, 2$ there is nothing to prove.

Suppose $l = -1$. If $y \in E_{\leq -1}(x)$ then $[x, y] = 0$ and $[x, [y, z]] = g_y(z)x + g_x(z)y$ for all $z \in L$. Moreover, as x is not a sandwich, there exists $u \in E_2(x)$. Now, as we have seen in the proof of Lemma 27, there is $v \in E_1(x)$ with $[x, v] \in E \cap (kx + ky)$. Consequently, the line $kx + ky = kx + k[x, v]$ is contained in $E \cap L_{\leq -1}(x)$. The converse follows from a straightforward verification that the defining equations for $y \in E_{-1}(x)$ are satisfied when $y = \alpha x + \beta[x, z]$ for $z \in L$ with $g_x(z) = 0$ and $\alpha, \beta \in k$.

It remains to consider the case $l = 0$. We must show that if $y \in E \cap N_L(x)$, then $y \in C_L(x)$. This is obvious if x is a scalar multiple of y, so suppose x and y are linearly independent. If $\alpha \in k$ is such that $[y, x] = \alpha x$, then $\alpha^2 x = [y, [y, x]] = g_y(x)y$, whence $\alpha = 0$, proving $y \in C_L(x)$.

(D). Suppose $(x, y) \in E_2$ and $z \in E_{\leq 0}(x)$. If $z \in E_{\leq -1}(y)$, then $[z, [y, u]] = g_y(u)z + g_z(u)y$ for each $u \in L$. As $g_z(x) = 0$, specialization to $u = x$ gives $[z, [y, x]] = g_y(x)z$. By assumption $g_y(x) \neq 0$ and by Jacobi $[z, [y, x]] = [[z, y], x] + [y, [z, x]] = 0$, so $z = 0$. Hence $E_{\leq 0}(x) \cap E_{\leq -1}(y) = \emptyset$, proving (D).

(E). Let $x \in E$. Since $\mathcal{E}_{\leq i}(kx)$ consists of the projective points of $E_{\leq i}(x) = E \cap L_{\leq i}(x)$ and $L_{\leq i}(x)$ are linear subspaces of L, it follows that the $\mathcal{E}_{\leq i}(kx)$ are subspaces of $(\mathcal{E}, \mathcal{F})$.

(F). This is immediate from the observation that $L_{\leq 1}(x) = g_x^{-1}(0)$ is a hyperplane of L.

(G). By the linear spanning of L by E, an isolated point of $(\mathcal{E}, \mathcal{E}_2)$ would be a sandwich.

As the lines of $(\mathcal{E}, \mathcal{F})$ are thick, by Lemma 5, every connected component of $(\mathcal{E}, \mathcal{E}_{-1})$ containing at least two points of $\mathcal{E}$ is a connected component of $(\mathcal{E}, \mathcal{E}_2)$ as well. These are the components which are non-degenerate root filtration spaces. By Lemma 7, the only possible relationship between $x, y \in \mathcal{E}$ from distinct components is $(x, y) \in \mathcal{E}_0$, whence the subalgebras generated (and, by Proposition 20, spanned) by different $\mathcal{B}_i$s commute. Thus L is indeed the direct sum of these subalgebras. □

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