Root shadow spaces

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Abstract

We give a characterization of the root shadow spaces of buildings whose types correspond to Dynkin diagrams. The results generalize earlier geometric point-line characterizations of certain spherical buildings as well as Timmesfeld's characterization of abstract root subgroups.

1 Introduction

In this paper, we characterize partial linear spaces that are root shadow spaces of spherical buildings. This means that the Coxeter type X_n of the building comes from a Dynkin diagram and that the shadow space has points that are flags whose types are the nodes J adjacent to the node extending the Dynkin diagram to an affine diagram. Such a shadow space is said to be of type $X_{n,J}$. For details, see Definition 3 of [6], where the characterization was announced and two applications were discussed. The only irreducible diagram X_n in which J is more than one node is A_n, where $J = \{1, n\}$ and the points of the corresponding root shadow space are the incident point-hyperplane pairs of a projective geometry of rank n.

Let $(\mathcal{E}, \mathcal{F})$ be a partial linear space. For $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$, a quintuple of symmetric relations partitioning $\mathcal{E} \times \mathcal{E}$, we call $(\mathcal{E}, \mathcal{F})$ a *root filtration space* with filtration $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$ if the following properties are satisfied, where we write $\mathcal{E}_{\leq i}$ for $\cup_{j\leq i}\mathcal{E}_j$.

(A) The relation $\mathcal{E}_{-2}$ is equality on $\mathcal{E}$.

(B) The relation $\mathcal{E}_{-1}$ is collinearity of distinct points of $\mathcal{E}$.

(C) There is a map $\mathcal{E}_1 \to \mathcal{E}$, denoted by $(u, v) \mapsto [u, v]$, such that, if $(u, v) \in \mathcal{E}_1$ and $x \in \mathcal{E}_i(u) \cap \mathcal{E}_j(v)$, then $[u, v] \in \mathcal{E}_{\leq i+j}(x)$.

(D) For each $(x, y) \in \mathcal{E}_2$, we have $\mathcal{E}_{\leq 0}(x) \cap \mathcal{E}_{\leq -1}(y) = \emptyset$.

(E) For each $x \in \mathcal{E}$, the subsets $\mathcal{E}_{\leq -1}(x)$ and $\mathcal{E}_{\leq 0}(x)$ are subspaces of $(\mathcal{E}, \mathcal{F})$.

(F) For each $x \in \mathcal{E}$, the subset $\mathcal{E}_{\leq 1}(x)$ is a geometric hyperplane of $(\mathcal{E}, \mathcal{F})$.

If, in addition, $(\mathcal{E}, \mathcal{F})$ satisfies the following two conditions, it is called a *non-degenerate* root filtration space.

(G) For each $x \in \mathcal{E}$ the set $\mathcal{E}_2(x)$ is not empty.

(H) The graph $(\mathcal{E}, \mathcal{E}_{-1})$ is connected.

Regarding relations $\mathcal{E}_i$, we adopt the terminology that became standard since

Long root group geometries, studied since the seventies, see [1, 7, 11], are examples of root filtration spaces, in which $\mathcal{E}_{\leq 0}$ corresponds to commuting, $\mathcal{E}_1$ to generating a special subgroup, and $\mathcal{E}_2$ to generating a subgroup isomorphic to a nontrivial quotient of $\mathrm{SL}(2, k)$ for some field k. Inspired by these early investigations, we adopt the following terminology. We call a pair $(x, z) \in \mathcal{E}_i$ *hyperbolic* (notation $x \overset{hyp}{\text{—}} z$) if $i = 2$, *special* (notation $x \overset{spec}{\text{—}} z$) if $i = 1$, *polar* (notation $x \overset{pol}{\text{—}} z$) if $i = 0$, *collinear* (notation $x \overset{coll}{\text{—}} z$) if $i = -1$ (so collinearity is only used for distinct points), and *commuting* (notation $[x, z] = 0$) if $i \leq 0$.

In [6] it is shown that, in a non-degenerate root filtration space, the filtration $(\mathcal{E}_i)_i$ is uniquely determined by the space $(\mathcal{E}, \mathcal{F})$. Therefore, we often drop the explicit reference to the filtration. Our main result reads as follows.

Theorem 1 *Let $(\mathcal{E}, \mathcal{F})$ be a non-degenerate root filtration space. If the singular rank of $(\mathcal{E}, \mathcal{F})$ is finite, then $(\mathcal{E}, \mathcal{F})$ is isomorphic to a shadow space of type* $\mathrm{A}_{n,\{1,n\}}$ *($n \geq 2$),* $(\mathrm{B}|\mathrm{C})_{n,2}$ *($n \geq 3$),* $\mathrm{D}_{n,2}$ *($n \geq 4$),* $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$, $\mathrm{F}_{4,1}$, *or* $\mathrm{G}_{2,2}$.

Here, the labeling of the nodes of the Coxeter diagrams follows [2], and the notation (B|C) is used to emphasize that we are dealing with Coxeter systems of type B rather than Dynkin systems.

The only root shadow spaces missing from this result are polar spaces, that is, the shadow spaces of type $(\mathrm{B}|\mathrm{C})_{n,1}$. A characterization of polar spaces in terms of degenerate root filtration spaces is given by Cuypers [8] who extended earlier results of Timmesfeld [11].

We proceed as follows. In section 2, we recall the basics of root filtration spaces from [6]. In section 3, we derive from the geometric axioms that the 'polar rank' of a non-degenerate root filtration space is either 2 (that is, the common neighborhood of a pair of points in $\mathcal{E}_0$ has no lines) or that we have a space satisfying the hypotheses of the Kasikova-Shult theorem of [9] (restated as Theorem 14 in this paper). Next, in section 4, we finish the proof of Theorem 1. We first deal with the polar rank 2 case. In Theorem 29, we find that if, in addition, each maximal singular subspace is a plane, the root filtration space is the Grassmannian of a polar space of rank 3, and in Theorem 23 we obtain that, otherwise, the root filtration space is the root shadow space of a projective geometry. This then characterizes all root shadow spaces except for polar spaces.

In section 5, we prove that the geometries satisfying Kasikova-Shult's hypotheses are also root filtration spaces (Proposition 39) and use this to show that root shadow spaces of buildings are root filtration spaces (Proposition 41).

2 Root filtration spaces

In this section we recall results of [6]. We begin with some notation for relations on a set $\mathcal{E}$. Let $x \in \mathcal{E}$. For a relation $\mathcal{X}$ on $\mathcal{E}$, we denote by $\mathcal{X}(x)$ the set of all elements $y \in \mathcal{E}$ with $(x, y) \in \mathcal{X}$. If, in addition, $y, z \in \mathcal{E}$ and $Y \subseteq \mathcal{E}$, we write $\mathcal{X}(x, y)$ for $\mathcal{X}(x) \cap \mathcal{X}(y)$, $\mathcal{X}(x, y, z)$ for $\mathcal{X}(x) \cap \mathcal{X}(y) \cap \mathcal{X}(z)$, and $\mathcal{X}(Y)$ for $\bigcap_{y \in Y} \mathcal{X}(y)$, etc.

A *point-line space* (or just *space*) is a pair $(\mathcal{E}, \mathcal{F})$ consisting of a set $\mathcal{E}$ (of points) and a collection $\mathcal{F}$ of subsets of $\mathcal{E}$ of size at least 2 (whose members are called lines). A space is called a *gamma space* if, for each point p and each line l not on p, the set of points on l collinear with p is either empty, a singleton, or all of l. It is called a *partial linear space* if every pair of distinct points is on at most one line. A *subspace* of $(\mathcal{E}, \mathcal{F})$ is a subset of $\mathcal{E}$ containing each line that has at least two points in common with it. The *rank* of a linear space is the length of a maximal chain of proper non-trivial subspaces; if there is no such chain, the rank is said to be ∞. A *singular subspace* of a space is a subspace in which any two points are collinear. The *singular rank* of a space is the supremum of all ranks of maximal singular subspaces.

We say that a subspace of a point-line space is a *geometric hyperplane*, or just *hyperplane*, if every line has a non-empty intersection with it. Thus, the whole point set is a geometric hyperplane. The definitions of *polar space*, as well as *non-degeneracy* and *rank* of a polar space, are as in [5].

We now focus on the axioms (A)–(F) of a root filtration space. In arguments, condition (C) applied to x will be referred as *filtration around* x. According to Lemma 2 below, $[u, v]$ is the unique point in $\mathcal{E}_{\leq -1}(u) \cap \mathcal{E}_{\leq -1}(v)$, so the map $[\cdot, \cdot]$ is uniquely determined by the relations $(\mathcal{E}_i)_i$.

We adopt the terminology of [11], referring to Condition (D) as the *triangle condition on* x, y, z. It is equivalent to the seemingly more general statement that, for each $(x, y) \in \mathcal{E}_2$, we have $\mathcal{E}_{\leq i}(x) \cap \mathcal{E}_{\leq j}(y) = \emptyset$ whenever $i + j < 0$.

Condition (E) can be replaced by the statement that $\mathcal{E}_{\leq i}(x)$ is a subspace of $(\mathcal{E}, \mathcal{F})$ for each i (see Lemma 2(i) below). The fact that $\mathcal{E}_{\leq -1}(x)$ is a subspace for every $x \in \mathcal{E}$ means that $(\mathcal{E}, \mathcal{F})$ is a gamma space.

For the remainder of this section, we assume that $(\mathcal{E}, \mathcal{F})$ is a root filtration space with filtration $(\mathcal{E}_i)_i$. We stress that (unless otherwise stated explicitly) the proofs of the following lemmas only use axioms (A)–(F).

Lemma 2 ([6], Lemma 1) *In* $(\mathcal{E}, \mathcal{F})$ *the following properties hold.*

(i) For each $i \in \{-2, \ldots, 2\}$ *and each* $x \in \mathcal{E}$*, the subset* $\mathcal{E}_{\leq i}(x)$ *is a subspace of* $(\mathcal{E}, \mathcal{F})$*.*

(ii) If $(u, v) \in \mathcal{E}_1$*, then* $[u, v]$ *is the unique common neighbor of both* u *and* v *in the collinearity graph* $(\mathcal{E}, \mathcal{E}_{-1})$ *of* $(\mathcal{E}, \mathcal{F})$*.*

(iii) If $(u, v) \in \mathcal{E}_1$*, then* $\mathcal{E}_0(u) \cap \mathcal{E}_2(v) \subseteq \mathcal{E}_1([u, v])$*.*

(iv) If $(x, y) \in \mathcal{E}_0$ *and* $z \in \mathcal{E}_{-1}(y)$*, then either* $z \in \mathcal{E}_{\leq 0}(x)$*, or* $z \in \mathcal{E}_1(x)$ *and* $\mathcal{E}_{-1}(x, y, z) = \{[x, z]\}$*.*

(v) If (x, q) *and* (u, z) *belong to* $\mathcal{E}_1$ *whereas* $u = [x, q]$ *and* $q = [u, z]$*, then* $(x, z) \in \mathcal{E}_2$*.*

(vi) If P *is a pentagon in the collinearity graph* $(\mathcal{E}, \mathcal{E}_{-1})$ *(that is, the induced subgraph is a pentagon), then each distinct non-collinear pair of points of* P *is polar.*

(vii) If $(u,v)\in\mathcal{E}_1$*, then* $\mathcal{E}_{-1}(u)\cap\mathcal{E}_0([u,v])\subseteq\mathcal{E}_1(v)$*.*

Lemma 3 ([6], Lemma 2) *The following three conditions are equivalent.*

(i) For each $(x,u)\in\mathcal{E}_{-1}$ *there exists a point in* $\mathcal{E}_2(x)\cap\mathcal{E}_1(u)$*.*

(ii) For each $(x,u)\in\mathcal{E}_{-1}$ *there exists a point in* $\mathcal{E}_{-1}(u)\cap\mathcal{E}_1(x)$*.*

(iii) For each $x\in\mathcal{E}$ *with* $\mathcal{E}_{-1}(x)\neq\emptyset$*, there exists a point in* $\mathcal{E}_2(x)$*.*

A line is said to be thick if it has at least three points, and a point-line space is called thick if each of its lines is thick.

Lemma 4 ([6], Lemma 3) *Assume that the conditions of Lemma 3 hold.*

(i) If $(\mathcal{E},\mathcal{F})$ *is thick and* $(\mathcal{E},\mathcal{E}_{-1})$ *is connected, then so is* $(\mathcal{E},\mathcal{E}_2)$*.*

(ii) If $(\mathcal{E},\mathcal{E}_2)$ *is connected and* $\mathcal{F}$ *is non-empty, then* $(\mathcal{E},\mathcal{E}_{-1})$ *is connected.*

Lemma 5 ([6], Lemma 4) *Suppose that* $(u,v)\in\mathcal{E}_1$ *and* $y\in\mathcal{E}_0([u,v],u,v)$*. If* $\mathcal{E}_{\leq -1}(v,y)\neq\emptyset$*, then* $\mathcal{E}_{-1}([u,v],y)\neq\emptyset$*.*

Lemma 6 ([6], Lemma 5) *Assume that* $(\mathcal{E},\mathcal{F})$ *satisfies (G). Then* $(\mathcal{E},\mathcal{F})$ *is the disjoint union of connected subspaces* $\mathcal{B}_i$ *such that* $\mathcal{B}_i\times\mathcal{B}_j\subseteq\mathcal{E}_0$ *whenever* $i\neq j$ *unless* $\mathcal{B}_i\times\mathcal{B}_j\subseteq\mathcal{E}_2$*, in which case* $\mathcal{B}_i$ *and* $\mathcal{B}_j$ *are singletons. Moreover, if* $x,y\in\mathcal{B}_i$ *for some* i *and* $(x,y)\in\mathcal{E}_0$ *then* $\mathcal{E}_{-1}(x,y)\neq\emptyset$*.*

Lemma 7 ([6], Lemma 6) *Suppose that* $(\mathcal{E},\mathcal{F})$ *is non-degenerate. Assume* $(x,y)\in\mathcal{E}_0$ *and* $u\in\mathcal{E}_{\leq -1}(x,y)$*. Then there exists* $v\in\mathcal{E}_{\leq -1}(x,y)$ *such that* v *is not collinear with* u*. In particular, every polar pair* (x,y) *is contained in a quadrangle.*

3 Parapolar spaces

In this section we assume that $(\mathcal{E},\mathcal{F})$ is a non-degenerate root filtration space. We shall first derive (Theorem 13) that it is a point-line space satisfying the axioms specified by Kasikova and Shult in [9]. Next, we apply the main result of [9] to derive a major step towards the proof of Theorem 1.

Lemma 8 *For $(x,y)\in\mathcal{E}_0$ the following two statements hold.*

(i) $\mathcal{E}_{-1}(x)\cap\mathcal{E}_1(y)\neq\emptyset$.

(ii) $\mathcal{E}_2(x)\cap\mathcal{E}_0(y)\neq\emptyset$.

Proof. (i). By Lemma 6 and (H), there is $u\in\mathcal{E}_{-1}(x,y)$. By Lemma 3, there exists $v\in\mathcal{E}_{-1}(x)\cap\mathcal{E}_1(u)$. By Lemma 2(vii), $y\in\mathcal{E}_1(v)$.

(ii). By (i), there is $u\in\mathcal{E}_{-1}(y)\cap\mathcal{E}_1(x)$. Set $v=[x,u]$. Then, by Lemma 2(ii),(iv), $v\in\mathcal{E}_{-1}(x,y,u)$. By (G) and Lemma 3, there exists $w\in\mathcal{E}_2(v)\cap\mathcal{E}_1(y)$. By (F) there is a point u' on the line vu such that $u'\overset{spec}{\text{---}}w$. Set $z=[w,u']$. We are going to show that $z\in\mathcal{E}_2(x)\cap\mathcal{E}_0(y)$. By filtration around y, we must have $z\in\mathcal{E}_{\le 0}(y)$. The triangle condition for v, z, w shows that $v\notin\mathcal{E}_{\le 0}(z)$. Since $u'\in\mathcal{E}_{\le -1}(v,z)$, we find $v\overset{spec}{\text{---}}z$ and hence $u'=[v,z]$. By Lemma 2(v) applied to the pairs (x,u') and (v,z) of $\mathcal{E}_1$, we obtain $z\in\mathcal{E}_2(x)$. From $y\in\mathcal{E}_{\le 0}(z)$ and the triangle condition for x, y, z, we conclude $y\in\mathcal{E}_0(z)$. $\Box$

Lemma 9 *Let $(x,z)\in\mathcal{E}_2$ and $(u,v)\in\mathcal{E}_{-1}\cup\mathcal{E}_0$ be such that $x\in\mathcal{E}_{-1}(u,v)$ and $z\in\mathcal{E}_1(u,v)$. Then, for $u'=[u,z]$ and $v'=[v,z]$, we have $(u,v')\in\mathcal{E}_0\cup\mathcal{E}_1$, $u'\neq v'$, and $[u',v']=0$. Moreover, one of the following two cases holds.*

(i) $u\overset{coll}{\text{---}}v$, $u'\overset{coll}{\text{---}}v'$, $u\overset{pol}{\text{---}}v'$, *and* $u'\overset{pol}{\text{---}}v$;

(ii) $u\overset{pol}{\text{---}}v$, $u'\overset{pol}{\text{---}}v'$, $u'\overset{spec}{\text{---}}v$, $u\overset{spec}{\text{---}}v'$, *and* $[u,v']=[u',v]$.

Proof. By the triangle condition on u,v,v' and on u,v,u', we find that neither $u\overset{hyp}{\text{---}}v'$ nor $v\overset{hyp}{\text{---}}u'$ holds. As $x\overset{spec}{\text{---}}v'$, the point $v=[x,v']$ is the unique element in $\mathcal{E}_{-1}(x,v')$ by Lemma 2(ii). This proves that neither $u\overset{coll}{\text{---}}v'$ nor $v\overset{coll}{\text{---}}u'$ holds. In particular, $(u,v')\in\mathcal{E}_0\cup\mathcal{E}_1$, and similarly for (v,u').

As $[x,u']=u\neq v=[x,v']$, we have $u'\neq v'$. The only alternative to $[u',v']=0$ would be $u'\overset{spec}{\text{---}}v'$ with $[u',v']=z$. As $u\in\mathcal{E}_{-1}(u')\cap\mathcal{E}_{\le 0}(v')$, the filtration around u would imply $[u,z]=0$, a contradiction.

(i). Suppose now $u \overset{coll}{\text{---}} v$. By Lemma 2(vi), the fact that z, v', v, u, u' cannot be a pentagon forces $u' \overset{coll}{\text{---}} v'$. Since u and v' are distinct points collinear with both u' and v, we find $u' \overset{pol}{\text{---}} v$, and similarly $u \overset{pol}{\text{---}} v'$.

(ii). Suppose next $u \overset{pol}{\text{---}} v$. If $u' \overset{pol}{\text{---}} v$, then Lemma 2(iv) gives $\mathcal{E}_{-1}(u', v, x) = \{u\}$, so $u \overset{coll}{\text{---}} v$, a contradiction. Hence $u' \overset{spec}{\text{---}} v$. Similarly, $u \overset{spec}{\text{---}} v'$. If $u' \overset{coll}{\text{---}} v'$, then as in (i), also $u \overset{coll}{\text{---}} v$, a contradiction. Therefore, $u' \overset{pol}{\text{---}} v'$. Write $y = [u, v']$. By Lemma 2(iv), $\mathcal{E}_{-1}(u, v, v') = \{y\}$, so $y \overset{coll}{\text{---}} v$. Similarly, y is collinear with u', so $y \in \mathcal{E}_{-1}(u', v)$. It follows that $y = [u', v]$. $\square$

We give an important consequence of Lemma 9 for the non-degenerate case.

Lemma 10 *For every pair $(v, w) \in \mathcal{E}_{-1}$, the set $\mathcal{E}_{\leq -1}(v) \cap \mathcal{E}_{\leq 0}(w)$ is a proper geometric hyperplane of $\mathcal{E}_{\leq -1}(v)$. In particular, if the points x, u, v span a singular plane of $(\mathcal{E}, \mathcal{F})$ and $w \in \mathcal{E}_1(x) \cap \mathcal{E}_{-1}(v)$, then there exists a point s on the line xu such that $w \overset{pol}{\text{---}} s$.*

Proof. To see the second assertion, use Lemma 3(ii) and pick $z \in \mathcal{E}_{-1}(w) \cap \mathcal{E}_1(v)$. Then, by Lemma 2(v), $x \overset{hyp}{\text{---}} z$ and, by (F), there is a point s on the line xu such that $z \overset{spec}{\text{---}} s$. Now $w = [z, v]$ and Lemma 9 gives $s \overset{pol}{\text{---}} w$.

As for the first statement, by Lemma 3(ii), the intersection $\mathcal{E}_{\leq -1}(v) \cap \mathcal{E}_1(w)$ is nonempty and hence $\mathcal{E}_{\leq -1}(v) \cap \mathcal{E}_{\leq 0}(w)$ is a proper subspace of $\mathcal{E}_{\leq -1}(v)$. To see that it is indeed a geometric hyperplane let xu be a line in $\mathcal{E}_{\leq -1}(v)$. If x does not lie in $\mathcal{E}_{\leq 0}(w)$, then the second assertion gives a point $s \in xu \cap \mathcal{E}_{\leq -1}(v) \cap \mathcal{E}_{\leq 0}(w)$. $\square$

Proposition 11 *Let $(x, y) \in \mathcal{E}_0$. Then $\mathcal{E}_{\leq -1}(x, y)$ is a subspace of $(\mathcal{E}, \mathcal{F})$ which is a non-degenerate polar space (possibly with an empty set of lines).*

Proof. By (H) and Lemma 6, $\mathcal{E}_{\leq -1}(x, y)$ is not empty, and by (E) it is a subspace of $(\mathcal{E}, \mathcal{F})$. We next establish the fundamental polar space axiom for this subspace. Let $u, v, w \in \mathcal{E}_{\leq -1}(x, y)$ be such that $u \overset{coll}{\text{---}} v$. By Lemma 8(ii), there exists $z \in \mathcal{E}_2(x) \cap \mathcal{E}_0(y)$. By the triangle condition on w, z, y, we must have $w \in \mathcal{E}_{\leq 1}(z)$. The triangle condition on x, z, w, then forces $w \overset{spec}{\text{---}} z$. Hence $w' = [w, z]$ exists; it satisfies $w' \overset{spec}{\text{---}} x$ and $w = [x, w']$. Furthermore, by the filtration around y, we have $y \overset{coll}{\text{---}} w'$. By Lemma 10, there is a point $s \in uv \cap \mathcal{E}_{\leq 0}(w')$. By the filtration around s (note that $x \overset{coll}{\text{---}} s$), we have $s \overset{coll}{\text{---}} [x, w'] = w$. This establishes that $\mathcal{E}_{\leq -1}(x, y)$ is a polar space. By Lemma 7, this polar space is non-degenerate. $\square$

Proposition 12 *Let $y \in \mathcal{E}$ and $l \in \mathcal{F}$ be such that $y \in \mathcal{E}_0(l)$. Then $\mathcal{E}_{\leq -1}(y,l)$ is a non-empty singular subspace of $(\mathcal{E},\mathcal{F})$.*

Proof. Suppose that a, b are non-collinear points of $\mathcal{E}_{\leq -1}(y,l)$. As $\mathcal{E}_{\leq -1}(a,b)$ has at least 2 points, $a \overset{pol}{\text{---}} b$, so $\mathcal{E}_{\leq -1}(a,b)$, which by Proposition 11 is a polar space containing both y and l, has a point in l collinear with y, contradicting the assumption $y \in \mathcal{E}_0(l)$ for every point x of l. Hence, $\mathcal{E}_{\leq -1}(y,l)$ is a singular subspace.

It remains to show that $\mathcal{E}_{\leq -1}(y,l)$ is non-empty. Pick $x \in l$. By Lemma 8(ii), there exists $z \in \mathcal{E}_2(x) \cap \mathcal{E}_0(y)$. Let v be the point of the line l such that $v \overset{spec}{\text{---}} z$, which exists by axiom (F), and set $v' = [v,z]$. Then $v \overset{coll}{\text{---}} v'$, $v' \overset{spec}{\text{---}} x$ and $v = [v',u]$ for every point u on the line l distinct from v. Furthermore, $[y,v'] = 0$ (as $[y,v] = [y,z] = 0$ by filtration around y). Assume $y \overset{coll}{\text{---}} v'$. Then, by the filtration around y we have $y \overset{coll}{\text{---}} v = [x,v']$, contradicting $y \in \mathcal{E}_0(l)$. Thus we actually have $y \overset{pol}{\text{---}} v'$.

Now, using Lemma 8(ii), pick $z' \in \mathcal{E}_2(v) \cap \mathcal{E}_0(y)$, let u be the point on the line l such that $z' \overset{spec}{\text{---}} u$, and put $u' = [z',u]$. As for v above, we see that $u' \in \mathcal{E}_{-1}(u) \cap \mathcal{E}_1(v) \cap \mathcal{E}_0(y)$. Observe that $u \neq v$, as $v \in \mathcal{E}_2(z')$ and $u \in \mathcal{E}_1(z')$. We also have $u' \overset{hyp}{\text{---}} v'$ by Lemma 2(v) and $\mathcal{E}_{\leq -1}(y,l) = \mathcal{E}_{\leq -1}(y,u,v)$ by axiom (E). Furthermore, from the fact that v is the only point of the line uv which is also in $\mathcal{E}_{\leq 1}(z)$ we infer $z \overset{hyp}{\text{---}} u$. For a point $w \in \mathcal{E}_{\leq -1}(u,y)$ the triangle condition on z, u, w gives $z \in \mathcal{E}_{\geq 1}(w)$ and the triangle condition on y, z, w gives $z \in \mathcal{E}_{\leq 1}(w)$, so $w \overset{spec}{\text{---}} z$. Hence, by Lemma 9, w is in $\mathcal{E}_{\leq -1}(v)$ as well if and only if $w \overset{pol}{\text{---}} v'$, and otherwise $w \overset{spec}{\text{---}} v'$.

By the argument of the first paragraph applied to uu' instead of the line l, using axiom (E) to see that uu' is contained in $\mathcal{E}_{\leq 0}(y)$, we find that $\mathcal{E}_{\leq -1}(y,u,u')$ is a singular subspace. Since $\mathcal{E}_{\leq -1}(y,u)$ is a non-degenerate polar space, there exists $w \in \mathcal{E}_{\leq -1}(y,u) \setminus \mathcal{E}_{\leq -1}(u')$.

Assume $\mathcal{E}_{\leq -1}(y,u,v) = \emptyset$. Then, since $y \in \mathcal{E}_0(v') \cap \mathcal{E}_{-1}(w)$, we have $v' \in \mathcal{E}_{\leq 1}(w)$ by the triangle condition. If $v' \overset{pol}{\text{---}} w$, then by Lemma 2(iv) applied to v', w and u we should have $v = [v',u] \in \mathcal{E}_{\leq -1}(v',w,u)$, contradicting $\mathcal{E}_{\leq -1}(y,u,v) = \emptyset$. Hence $w \overset{spec}{\text{---}} v'$. For $w' = [w,v']$, by the filtration around y, we have $y \overset{coll}{\text{---}} w'$. Also, if $v \overset{spec}{\text{---}} w$ then $u = [v,w]$, and the filtration around y would imply $y \overset{coll}{\text{---}} u = [v,w]$, a contradiction. Therefore, $v \overset{pol}{\text{---}} w$ and hence, by the filtration around v, we find $v \overset{coll}{\text{---}} w'$. As we also have $v' \overset{coll}{\text{---}} w'$, we see that $[w,v'] = w' \in \mathcal{E}_{\leq -1}(y,v,v')$.

Similarly, we find that $u' \overset{spec}{\text{---}} w'$ and $t := [w',u'] \in \mathcal{E}_{\leq -1}(y,u,u')$. Since $w \notin \mathcal{E}_{-1}(u')$, the points t and w are distinct. Obviously, $w' \overset{coll}{\text{---}} t$ and $w \overset{coll}{\text{---}} w'$.

If $w \overset{coll}{\text{---}} t$ as well, then by Lemma 10 there is a point s on the line wt such that $s \overset{pol}{\text{---}} v'$ whence $s \overset{coll}{\text{---}} v$, contrary to assumption. Since u and w' are distinct points of $\mathcal{E}_{\leq -1}(w,t)$, we must have $w \overset{pol}{\text{---}} t$. Now u is a point and yw' is a line of the polar space $\mathcal{E}_{\leq -1}(w,t)$ whence there is a point s on the line yw' collinear with u. We have $s \in \mathcal{E}_{\leq -1}(u) \cap \mathcal{E}_{\leq 0}(v')$ so, the filtration around s gives $s \overset{coll}{\text{---}} v$. This implies $s \in \mathcal{E}_{\leq -1}(y,u,v)$, a final contradiction. □

We recall some definitions from [5]. A *parapolar space* is a connected partial linear gamma space possessing a collection of geodesically closed subspaces, called *symplecta* (singular: *symplecton*), isomorphic to non-degenerate polar spaces of rank at least 2, with the properties that each line is contained in a symplecton and that each pair of distinct non-collinear points having at least 2 common neighbors is contained in a unique symplecton. In this definition, we have taken the opportunity to correct an error in [5], where the existence of symplecta was only required for quadrangles (rather than any set of points at mutual distance 2 on a proper 4 circuit that is not necessarily a quadrangle). If all symplecta are polar spaces of rank k (respectively, of rank at least k) the space is said to have *polar rank* k (respectively, polar rank at least k).

With [9] in mind, we call a partial linear space a *root parapolar space* if it is a parapolar space satisfying the following two axioms.

(a) For each point x and each symplecton S the set of points of S collinear with x is either empty or contains a line.

(b) For each singular plane π and each line l meeting π at a point x and not lying in a singular subspace containing π, either one or all lines m on x in π have the property that $l \cup m$ belongs to a symplecton.

In addition, a root parapolar space is called *non-degenerate* if it satisfies the following axiom.

(c) For each pair x, y of distinct collinear points, there is a point z collinear with x but not collinear with y such that x is the only point collinear with both y and z.

Note that in [9] condition (b) for root parapolar spaces is stated in a slightly stronger form:

(b') For each singular plane π and each line l meeting π at a point x, either one or all lines m on x in π have the property that $l \cup m$ belongs to a symplecton.

However, as in a parapolar space of polar rank at least 3 every singular plane is contained in a symplecton, conditions (b) and (b') are equivalent in our setting.

Theorem 13 *Let $(\mathcal{E}, \mathcal{F})$ be a non-degenerate root filtration space. If some line in $\mathcal{F}$ is contained in a unique maximal singular subspace, then so is every line in $\mathcal{F}$. Otherwise $(\mathcal{E}, \mathcal{F})$ is a non-degenerate root parapolar space of polar rank at least* 3.

Proof. Assume that the line m is contained in two planes P and Q which do not span a singular subspace. Let x be a point of m.

We first show that there is $y \in \mathcal{E}_0(x)$ such that the polar rank of $\mathcal{E}_{\leq -1}(x, y)$ is at least 2. Pick $z \in \mathcal{E}_2(x)$ and $u \in P \setminus m$ and $v \in Q \setminus m$ such that $z \in \mathcal{E}_1(u, v)$. We are in the case $u \overset{pol}{\text{—}} v$ (because $m \subseteq \mathcal{E}_{\leq -1}(u, v)$) and hence, by Lemma 9, $u \overset{spec}{\text{—}} [z, v]$. Setting $y = [u, [z, v]]$, we find $y \in \mathcal{E}_{\leq -1}(u, v)$. Note that $y \overset{coll}{\text{—}} x$ would imply that both u and y belong to $\mathcal{E}_{\leq -1}([u, z], x)$, so $u = y$, whence $u \overset{coll}{\text{—}} v$, a contradiction. Therefore, $x \overset{pol}{\text{—}} y$. By Proposition 11, there is $w \in m \cap \mathcal{E}_{\leq -1}(y)$; now uw is a line contained in $\mathcal{E}_{\leq -1}(x, y)$, so the latter space has polar rank at least 2, as claimed.

Now let l be a line which intersects m in the point x. If l intersects $\mathcal{E}_{\leq -1}(x, y)$ then, as $\mathcal{E}_{\leq -1}(x, y)$ is a non-degenerate polar space of polar rank at least 2, the line l is contained in two planes that do not span a singular subspace.

If l does not intersect $\mathcal{E}_{\leq -1}(x, y)$ then either $l \subseteq \mathcal{E}_0(y)$ and, by Proposition 12, $\mathcal{E}_{\leq -1}(y, l)$ must be a non-empty singular subspace, or there is a point in $l \cap \mathcal{E}_1(y)$, whose commutator with y belongs to $\mathcal{E}_{-1}(l, y)$ by Lemma 2(iv). In each case, there exists $t \in \mathcal{E}_{\leq -1}(y, l)$. As $\mathcal{E}_{\leq -1}(x, y)$ is a polar space of rank at least 2, the point t is contained in a quadrangle of $\mathcal{E}_{\leq -1}(x, y)$. But $\mathcal{E}_{\leq -1}(y, l)$ is a singular subspace, so there is a point s of $\mathcal{E}_{\leq -1}(x, y, t)$ which is not in $\mathcal{E}_{\leq -1}(l)$. Now we replace m with the line xt, P with the plane lt and Q with xts and apply the first part of the proof. To be more specific, in this case $z \in \mathcal{E}_2(x)$ as above, u is some point on l, and v is some point on xs such that $z \in \mathcal{E}_1(u, v)$. We obtain that $y = [u, [z, v]]$ is a point in $\mathcal{E}_0(x)$ such that $\mathcal{E}_{-1}(x, y)$ is a polar space of rank at least 2 and now l intersects $\mathcal{E}_{-1}(x, y)$ in

u (and l still intersects m in x). It follows that l is contained in two planes that do not span a singular subspace.

Thus the property of m is inherited by any line intersecting m and consequently to any line. We have seen that if there is a line contained in two planes not in a common singular subspace then every line is. Finally, if there are two distinct maximal subspaces containing a line l then we can choose a pair of non-collinear points a and b of these subspaces and these points together with the line l, contained in both subspaces, span two planes which intersect in a line and do not span a singular subspace.

An even stronger statement can be proved in this case: Assume that the point $x \in \mathcal{E}$ and the line $l \in \mathcal{F}$ span a singular plane. Then there exists a point $y \in \mathcal{E}_0(x)$ such that y and l also span a singular plane. Indeed, pick a and b as above. As $l \subseteq \mathcal{E}_{-1}(a, b, x)$, the points a and b belong to $\mathcal{E}_{\leq 0}(x)$. If $x \notin \mathcal{E}_{-1}(a, b)$ then either a or b will be a good choice for y. Otherwise as l is a line in the non-degenerate polar space $\mathcal{E}_{-1}(a, b)$, it coincides with the set of points of $\mathcal{E}_{-1}(a, b)$ that are collinear with every point collinear with all of the points of l. This implies the existence of a point $y \in \mathcal{E}_{-1}(a, b) \setminus \mathcal{E}_{\leq -1}(x)$.

Suppose now that every line is contained in at least two distinct maximal singular subspaces. If x and y are two points of $\mathcal{E}$ at mutual distance 2, then, by Lemma 2 (ii) and Lemma 7, either $x \overset{spec}{\text{---}} y$ and $\mathcal{E}_{-1}(x, y) = \{[x, y]\}$ or $x \overset{pol}{\text{---}} y$ and $\mathcal{E}_{-1}(x, y)$ is a non-degenerate polar space.

We claim that, in the polar case, the polar space $\mathcal{E}_{-1}(x, y)$ has polar rank at least 2. To see this, pick $z \in \mathcal{E}_2(x) \cap \mathcal{E}_0(y)$ (it exists by Lemma 8). By Lemma 7 there are $u, v \in \mathcal{E}_{-1}(x, y)$ with $u \overset{pol}{\text{---}} v$. By the triangle condition, $u, v \in \mathcal{E}_1(z)$ and, by Lemma 9, $v' = [v, z] \in \mathcal{E}_1(u)$. Also, $v' \in E_{-1}(y)$ by the filtration around y. Together with $y \in E_{-1}(u)$ this implies $y = [u, v']$. By the strong property proved two paragraphs above there exists a point $w' \in \mathcal{E}_{-1}(v, v') \cap \mathcal{E}_0(y)$. Lemma 10 implies that there is a point w on the line $v'w'$ which is also in $\mathcal{E}_0(x)$. Since $v' \overset{spec}{\text{---}} x$, the point w is distinct from v' and hence $w \in \mathcal{E}_0(y)$. In view of the triangle condition on w, y, u, we have $w \in \mathcal{E}_{\leq 1}(u)$. If w were in $\mathcal{E}_{\leq 0}(u)$ then the filtration around w would give $y = [u, v'] \in \mathcal{E}_{\leq -1}(w)$, a contradiction with $y \in \mathcal{E}_0(w)$. Thus $w \in \mathcal{E}_1(u)$. By the filtration around x and y, respectively, we have $[w, u] \in \mathcal{E}_{-1}(x, y)$. As the subspace $\mathcal{E}_{-1}(x, y)$ contains the line $u[w, u]$, it has polar rank at least 2. This proves the claim.

By Cooperstein's theory, see [4], this claim, together with (E), implies that $(\mathcal{E}, \mathcal{F})$ is a parapolar space with symplecta of rank at least 3.

Continuing with the case where every line is contained in at least 2 distinct maximal singular subspaces, we show that axiom (a) for root parapolar spaces holds. Assume that S is a symplecton of $(\mathcal{E},\mathcal{F})$ containing a point x collinear to a point z outside S. Then $\mathcal{E}_{\leq -1}(z)\cap S$ contains x. Choose a point $y\in S$ not collinear with x. Then $z\in\mathcal{E}_{\leq 1}(y)$ by the triangle condition. If $z\in\mathcal{E}_1(y)$, then $[y,z]$ belongs to $\mathcal{E}_{-1}(x,y,z)$ by Lemma 2(iv). So $[y,z]\in S$ by convexity of S and z is collinear to all points on the line $x[y,z]\subseteq S$. If $z\in\mathcal{E}_{\leq 0}(y)$, then by (E) every point of the line xz belongs to $\mathcal{E}_{\leq 0}(y)$, so either there is a point $u\in xz\cap\mathcal{E}_{\leq -1}(y)$ and $z\in xz=xu\subseteq S$, or each point of xz belongs to $\mathcal{E}_0(y)$ and, by Proposition 12, $\mathcal{E}_{\leq -1}(x,y,z)\neq\emptyset$. The first case is a contradiction with $z\notin S$, and the second case leads to (a).

Axiom (b) follows directly from Lemma 10 and axiom (c) is a direct consequence of (G) and Lemma 3. Hence $(\mathcal{E},\mathcal{F})$ is a non-degenerate root parapolar space. $\square$

In case the polar rank is at least 3, the main result of [9] applies. It reads as follows.

Theorem 14 (Kasikova & Shult) *Each root parapolar space of polar rank at least* 3 *satisfying the following condition (d) is a shadow space of type* $(\mathrm{B}|\mathrm{C})_{n,2}$ $(n\geq 4)$, $\mathrm{D}_{n,2}$ $(n\geq 4)$, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$, *or* $\mathrm{F}_{4,1}$.

(d) If all symplecta have rank at least 4*, then every singular subspace has finite rank.*

Suppose now that $(\mathcal{E},\mathcal{F})$ is a non-degenerate root filtration space with finite singular rank. If a line is contained in at least 2 maximal singular subspaces, then, by Theorem 13, $(\mathcal{E},\mathcal{F})$ is a non-degenerate root parapolar space of polar rank at least 3, so Theorem 14 applies and $(\mathcal{E},\mathcal{F})$ is isomorphic to one of the spaces listed in the conclusion of Theorem 1. The other case will be dealt with in the next section.

4 Polar rank 2

In this section, we finish the proof of Theorem 1. More precisely, we establish the following result, which involves the spaces $\mathcal{E}(\mathbb{P},\mathbb{H})$ of [6]. For convenience of the reader, we repeat its definition here. Let $\mathbb{P}$ be a projective space and let $\mathbb{H}$ be a collection of hyperplanes forming a subspace of the dual of $\mathbb{P}$

annihilating $\mathbb{P}$. The latter means that the intersection of all hyperplanes of $\mathbb{H}$ is empty. If $\mathbb{P}$ has finite rank, this condition forces $\mathbb{H}$ to be the dual of $\mathbb{P}$. Take $\mathcal{E}$ to be the set of incident pairs from $\mathbb{P} \times \mathbb{H}$. The set $\mathcal{F}$ of lines is built up of two kinds: those consisting of all (x, H) with hyperplane $H \in \mathbb{H}$ fixed and x running through the points of a line of $\mathbb{P}$ inside H, and those consisting of all (x, H) with point x fixed and H running through the hyperplanes in $\mathbb{H}$ containing a fixed codimension 2 subspace of $\mathbb{P}$ containing x. So, $((x, H), (y, K)) \in \mathcal{E}_{-1}$ iff $x = y$ or $H = K$ (but not both). Then $(\mathcal{E}, \mathcal{F})$ is a root filtration space with $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$ defined as follows: $(x, H) \in \mathcal{E}_0((y, K))$ iff $x \in K$ and $y \in H$ and $x \neq y$ and $H \neq K$, $(x, H) \in \mathcal{E}_1((y, K))$ iff $x \in K$ (in which case $[(x, H), (y, K)] = (x, K)$) or $y \in H$ (in which case $[(x, H), (y, K)] = (y, H)$) but not both, and $(x, H) \in \mathcal{E}_2((y, K))$ iff $x \notin K$ and $y \notin H$. We denote this root filtration space by $\mathcal{E}(\mathbb{P}, \mathbb{H})$.

We do not need a finiteness assumption on singular ranks. Recall from Proposition 11 that $\mathcal{E}_{\leq -1}(x, y)$ is a non-degenerate polar space.

Theorem 15 *Suppose that $(\mathcal{E}, \mathcal{F})$ is a non-degenerate root filtration space such that every line $l \in \mathcal{F}$ is contained in a unique maximal singular subspace. Then either $(\mathcal{E}, \mathcal{F})$ is a shadow space of type* $(\mathrm{B}|\mathrm{C})_{3,2}$ *or* $\mathrm{G}_{2,2}$*, or there is a projective space $\mathbb{P}$ and a collection of hyperplanes $\mathbb{H}$ of $\mathbb{P}$ forming a subspace of the dual of $\mathbb{P}$ and annihilating $\mathbb{P}$ such that $\mathcal{E}$ is isomorphic to $\mathcal{E}(\mathbb{P}, \mathbb{H})$.*

In particular, if the singular rank of $(\mathcal{E}, \mathcal{F})$ is finite, then $(\mathcal{E}, \mathcal{F})$ is a shadow space of type $\mathrm{A}_{n,\{1,n\}}$, $(\mathrm{B}|\mathrm{C})_{3,2}$, *or* $\mathrm{G}_{2,2}$.

Thus, let $(\mathcal{E}, \mathcal{F})$ be a non-degenerate root filtration space. Throughout this section, we assume that every line $l \in \mathcal{F}$ is contained in a unique maximal singular subspace, which we denote by $M(l)$. This assumption implies that for $(x, y) \in \mathcal{E}_0$, the subspace $\mathcal{E}_{\leq -1}(x, y)$ has no lines; in other words, $(\mathcal{E}, \mathcal{F})$ has polar rank 2. In Proposition 17, we shall see that any two maximal singular subspaces of $(\mathcal{E}, \mathcal{F})$ have equal rank. If the singular rank of $(\mathcal{E}, \mathcal{F})$ equals 1, then clearly $(\mathcal{E}, \mathcal{F})$ is a generalized hexagon, whence a shadow space of type $\mathrm{G}_{2,2}$; this is the content of Corollary 18. Therefore, except for Lemma 16 and Proposition 17, throughout this section, we assume that the singular rank of $(\mathcal{E}, \mathcal{F})$ is at least 2, that is, l is a proper subspace of $M(l)$.

It will be convenient to use the following notation. Let X be a subset of $\mathcal{E}$ and Y a subset or an element of $\mathcal{E}$. Then $C_X(Y)$ stands for the centralizer of Y in X, i.e., $C_X(Y) = X \cap \mathcal{E}_{\leq 0}(Y)$. By Axiom (E), if X is a subspace of $(\mathcal{E}, \mathcal{F})$ then $C_X(Y)$ is a subspace as well.

4.1 General properties

Besides $G_{2,2}$, we need to find two more types of point-line spaces of the conclusion of Theorem 15, viz. $A_{n,\{1,n\}}$ and $(B|C)_{3,2}$. In this subsection we show how to distinguish between these two cases. In the next two subsections (Theorems 35 and 29), we will deal with the individual cases.

Lemma 16 *Let M be a maximal singular subspace of $(\mathcal{E},\mathcal{F})$ and $v \in \mathcal{E} \setminus M$. Then exactly one of the following holds.*

- *(i) $M \cap \mathcal{E}_1(v)$ is a proper hyperplane of M and $M \subseteq \mathcal{E}_1(v) \cup \mathcal{E}_2(v)$.*
- *(ii) $C_M(v)$ is a proper hyperplane of M, there is a unique point u of M such that $\mathcal{E}_{-1}(v) \cap M = \{u\}$ and $M \subseteq \mathcal{E}_{\leq 1}(v)$.*

Proof. Obviously, case (i) holds if and only if $\mathcal{E}_2(v) \cap M \neq \emptyset$.

We claim that if there exists a line ab inside M such that $u \overset{\mathit{spec}}{\text{---}} v$ for every $u \in ab$, then there is a point $x \in M$ such that $x \overset{\mathit{hyp}}{\text{---}} v$. To see this set $a' = [a, v]$ and $b' = [b, v]$. Then $[a', b] = 0$ by the filtration around b and either $b' \overset{\mathit{coll}}{\text{---}} a'$ or $b' = a'$ by the filtration around a'. If $b' = a'$ then by Lemma 10 there is a point in $C_{ab}(v)$, contrary to assumption. Thus $a' \overset{\mathit{coll}}{\text{---}} b'$. We also have $a' \overset{\mathit{pol}}{\text{---}} b$ since $a' \overset{\mathit{coll}}{\text{---}} b$ would lead to $a' = b'$. By Lemma 8(ii) there exists $z \in \mathcal{E}_2(a') \cap \mathcal{E}_0(b)$. Then $z \overset{\mathit{spec}}{\text{---}} a$. Set $x = [z, a]$. By the filtration around b, we have $x \overset{\mathit{coll}}{\text{---}} b$. As also $x \overset{\mathit{coll}}{\text{---}} a$, the point x belongs to M, the unique maximal singular subspace containing ab. The point x cannot commute with a' (by the triangle condition for z, x, a') and $x \overset{\mathit{hyp}}{\text{---}} a'$ is also excluded (by the triangle condition for a, x, a'). Thus $a' \overset{\mathit{spec}}{\text{---}} x$ and $a = [a', x]$. Finally, x is hyperbolic to v by Lemma 2(v) applied to the path x, a, a', v. We have proved the claim.

Assume that case (i) does not hold, so $\mathcal{E}_2(v) \cap M = \emptyset$. By the claim and axiom (E), $C_M(v)$ is a hyperplane of M, possibly all of M. Assume that there exists a point $u \in M$ such that $u \overset{\mathit{coll}}{\text{---}} v$. (By uniqueness of M on any line, there is at most one such point.) By Lemma 3(ii), there is $v' \in \mathcal{E}_{-1}(v) \cap \mathcal{E}_1(u)$ (and hence $v = [u, v']$). If there is $w \in \mathcal{E}_{\leq 0}(v') \cap M$, then the filtration around w gives $w \overset{\mathit{coll}}{\text{---}} [u, v'] = v$, whence $w = u$ and $w \notin \mathcal{E}_{\leq 0}(v')$, a contradiction. Thus v' does not commute with any point of M and hence, by the claim, there exists $x \in M \cap \mathcal{E}_2(v')$. By the triangle condition for x, v, v', the points x and v cannot commute. Consequently, $C_M(v)$ is indeed a proper hyperplane of M.

Finally assume there exists $u \in C_M(v)$. In view of the above it is sufficient to show that M also contains a point collinear with v. Pick $w \in \mathcal{E}_{\leq -1}(u, v)$. Then, by the previous discussion, there exists a point $x \in M$ such that $x \overset{spec}{\text{—}} w$ and $u = [x, w]$. If v and x commute then, by the filtration around v, we have $v \overset{coll}{\text{—}} [w, x] = u$, contrary to assumption. Thus $x \overset{spec}{\text{—}} v$. Also, by the filtration around u, we have $u \overset{coll}{\text{—}} [x, v]$. But $x \overset{coll}{\text{—}} [x, v]$ as well, so $[x, v]$ is in $M(ux) = M$. As $[x, v]$ is also collinear with v, it is a point as required. □

Proposition 17 *Let M_1 and M_2 be two maximal singular subspaces of $(\mathcal{E}, \mathcal{F})$ containing the point z. Then the maps $C_{M_2} : 2^{M_1} \to 2^{M_2}$ and $C_{M_1} : 2^{M_2} \to 2^{M_1}$ give a Galois connection between the power sets 2^{M_1} and 2^{M_2} where the closed sets are subspaces of M_i containing z. Every subspace of M_i of finite rank containing z is closed. As a consequence, the ranks of maximal singular subspaces of $(\mathcal{E}, \mathcal{F})$ are equal.*

Proof. It is obvious that closed subsets are indeed subspaces containing z. Let $\mathcal{P} = M_1 \cup M_2$ and $\mathcal{L} = (\mathcal{F} \cap (2^{M_1} \cup 2^{M_2})) \cup \{\{u, v\} \mid u \in M_1, v \in C_{M_2}(u)\}$. By Lemma 16, for every $u \in M_1 \setminus \{z\}$, the centralizer $C_{M_2}(u)$ is a proper hyperplane of M_2 and $C_{M_1}(v)$ is a proper hyperplane of M_1 for every $v \in M_2 \setminus \{z\}$, so the point-line space $(\mathcal{P}, \mathcal{L})$ is a polar space with radical $\{z\}$. In the non-degenerate polar space $(\mathcal{P}, \mathcal{L})/\{z\}$, for every singular subspace X of finite rank we have $X = X^{\perp\perp}$, where $Y^\perp$ for a set of points Y, is the set of all points of the space collinear with each point of Y (cf. [5], Theorems 2.4 and 3.1). For every subspace X of M_i containing z, this implies $X = C_{\mathcal{P}}(C_{\mathcal{P}}(X))$, proving that X is closed indeed.

In view of the Galois connection, two intersecting maximal singular subspaces have equal rank. By connectedness of $(\mathcal{E}, \mathcal{F})$ this implies that all maximal singular subspaces of $(\mathcal{E}, \mathcal{F})$ have the same rank. □

Corollary 18 *If there is a line in $\mathcal{F}$ which is a maximal singular subspace of $(\mathcal{E}, \mathcal{F})$ then $(\mathcal{E}, \mathcal{F})$ is a generalized hexagon, i.e., a shadow space of type $\mathrm{G}_{2,2}$.*

Proof. By the proposition, the singular rank of $(\mathcal{E}, \mathcal{F})$ equals 1. Emptiness of $\mathcal{E}_0$ follows from Lemma 16. □

From now on we assume that the singular rank of $(\mathcal{E}, \mathcal{F})$ is at least 2.

Lemma 19 *For any three points x, y, z of a singular subspace of $\mathcal{E}$ not all on one line, the following assertions hold.*

(i) $\mathcal{E}_1(x) \cap \mathcal{E}_0(y) \cap \mathcal{E}_{-1}(z) \neq \emptyset$.

(ii) *The union of all lines* zv *for* $v \in xy$ *is a subspace of* $\mathcal{E}$.

(iii) *Every singular subspace of* $\mathcal{E}$ *is projective.*

Proof. Let M_1 be the maximal singular subspace containing x, y, z and let M_2 be another one through z. Then, by Proposition 17, $yz = C_{M_1}(C_{M_2}(y))$ (this shows (i)) and $\cup_{v \in xy} zv = C_{M_1}(C_{M_2}(\{x, y\}))$ (this shows (ii)). To see (iii) let x, y, z be three non-collinear points of some maximal singular subspace, say M_1. Then, by (ii), the subspace spanned by x, y, and z is $P = \cup_{v \in xy} zv = \cup_{v \in xz} yv = \cup_{v \in yz} xv$. We have to prove that P is a projective plane. Let l be a line of P. If l contains z then by (i) there exists a point $u \in \mathcal{E}$ such that $u \overset{coll}{\rule{1em}{0.4pt}} z$ and $C_{xy}(u) = l \cap xy$, whence $l = C_P(u)$. As $C_{M_1}(u)$ is a hyperplane of M_1, its intersection $C_P(u) = l$ with P must also be a hyperplane of P. If $z \notin l$ let v, w be two distinct points of l. Then by (i) there exists $u \in \mathcal{E}$ such that $u \overset{coll}{\rule{1em}{0.4pt}} v$, $u \overset{pol}{\rule{1em}{0.4pt}} w$, and $u \overset{spec}{\rule{1em}{0.4pt}} z$. As $C_{M_1}(u)$ is a hyperplane of M_1 and $z \notin \mathcal{E}_{\leq 0}(u)$, the subspace $C_P(u)$ is a proper hyperplane of P. Obviously $l \subseteq C_P(u)$. Assume that there exists a point $p \in C_P(u) \setminus l$. Then $p \neq z$ and, as we have seen, the line pz is a hyperplane of P whence l intersects pz. Now pz contains two distinct points of $\mathcal{E}_{\leq 0}(u)$, so $pz \subseteq \mathcal{E}_{\leq 0}(u)$; in particular $z \in \mathcal{E}_{\leq 0}(u)$, a contradiction. Therefore, $l = C_P(u)$ is again a hyperplane of P. We have shown that every line of P is a hyperplane of P, whence (iii). □

Lemma 20 *If* $(u, v) \in \mathcal{E}_0$ *and* $x \in \mathcal{E}_{\leq -1}(u, v)$ *then* $\mathcal{E}_{\leq 0}(u, v) \subseteq \mathcal{E}_{\leq 0}(x)$. *If, moreover,* $y \in \mathcal{E}_{\leq -1}(u, v)$ *is distinct from* x, *then* $(x, y) \in \mathcal{E}_0$ *and* $\mathcal{E}_{\leq 0}(u, v) = \mathcal{E}_{\leq 0}(x, y)$.

Proof. Assume $w \in \mathcal{E}_{\leq 0}(u, v) \setminus \mathcal{E}_{\leq 0}(x)$. Then the only possibility for the relation of x and w is $x \overset{spec}{\rule{1em}{0.4pt}} w$ (triangle xvw). Set $y = [x, w]$. The filtration around u gives $y \in \mathcal{E}_{\leq -1}(u)$. Similarly, $y \in \mathcal{E}_{\leq -1}(v)$. The relation $u \overset{pol}{\rule{1em}{0.4pt}} v$ excludes the equalities $y = u$ and $y = v$, so $y \in \mathcal{E}_{-1}(u, v)$. From this we see that the line xy is contained in the planes xyu and xyv. Since these planes do not generate a singular subspace, we have reached a contradiction. This establishes the first assertion.

Suppose now y is as stated. Then, by the first assertion, $\mathcal{E}_{\leq 0}(u, v) \subseteq \mathcal{E}_{\leq 0}(x, y)$. Clearly, $(x, y) \in \mathcal{E}_{\leq 0}$, so the assumptions on y and on the polar rank being 2 imply $(x, y) \in \mathcal{E}_0$. We can therefore apply the first assertion to x and y instead of u and v to obtain $\mathcal{E}_{\leq 0}(x, y) \subseteq \mathcal{E}_{\leq 0}(u, v)$. □

Lemma 21 *Assume $(x, y) \in \mathcal{E}_0$ and $u, v \in \mathcal{E}_{\leq -1}(x, y)$ with $u \neq v$. Let y' be a point of the line yu. Then there exists a (unique) point v' on the line vx such that $y' \overset{coll}{\text{—}} v'$.*

Proof. Set $M_1 = M(xu)$ and $M_2 = M(xv)$. By Lemma 20, $C_{M_2}(\{u, v\}) = C_{M_2}(\{x, y\})$, so $C_{M_2}(u) = C_{M_2}(\{u, v\}) = C_{M_2}(\{x, y\}) = C_{M_2}(y)$, whence $xu = C_{M_1}(C_{M_2}(u)) = C_{M_1}(C_{M_2}(y))$. By symmetry, we also have $xv = C_{M_2}(C_{M_1}(y))$.

If $y' = u$ then $v' = x$. If $y' \neq u$ then $C_{M_1}(y') = C_{M_1}(y)$. Let v' be the unique point of M_2 collinear with y'. Then the preceding argument, applied to x, y', u and v' shows that $xv' = C_{M_2}(C_{M_1}(y')) = C_{M_2}(C_{M_1}(y)) = xv$. Therefore v' must be on the line xv. $\square$

Lemma 22 *There is no quadruple $x, u, v, w \in \mathcal{E}$ such that $x \overset{coll}{\text{—}} u$, $x \overset{coll}{\text{—}} v$, $x \overset{coll}{\text{—}} w$, $u \overset{spec}{\text{—}} v$, $u \overset{pol}{\text{—}} w$, and $v \overset{pol}{\text{—}} w$.*

Proof. Suppose the contrary. Pick $y \in \mathcal{E}_{\leq -1}(u, w) \setminus \{x\}$. Then $y \overset{hyp}{\text{—}} v$ is excluded by the triangle yvw. On the other hand, in the case $[y, v] = 0$ from the filtration around y we would infer $x \overset{coll}{\text{—}} y$ and hence $\mathcal{E}_{\leq -1}(u, w)$ would contain the line xy, a contradiction with the existence of only one maximal singular subspace containing a line. Thus we have $v \overset{spec}{\text{—}} y$. Also, $x \overset{pol}{\text{—}} y$ since $u, w \in \mathcal{E}_{\leq -1}(x, y)$. Let $z = [v, y]$. By the filtration around x we have $x \overset{coll}{\text{—}} z$ or $x = z$. The second case cannot happen as x is not collinear with y, so $x \overset{coll}{\text{—}} z$. A similar argument shows $z \overset{coll}{\text{—}} w$. But then the line xz is in $\mathcal{E}_{\leq -1}(w, v)$, a contradiction. $\square$

Proposition 23 *Suppose that $(\mathcal{E}, \mathcal{F})$ is a non-degenerate root filtration space in which each line is contained in exactly one singular subspace, which is of rank at least 2. If there is a point of $\mathcal{E}$ which is contained in at least 3 maximal singular subspaces, then $(\mathcal{E}, \mathcal{F})$ has singular rank 2.*

Proof. Let x be a point belonging to at least 3 maximal singular subspaces. Assume that l is a line on x that is not a hyperplane of $M(l)$. By Lemma 19(i), we can find $u \in \mathcal{E}_{-1}(x) \cap \mathcal{E}_0(l \setminus \{x\})$. Let M' be a maximal singular subspace containing x not containing l and xu. By Lemma 16 there exists a point $v \in M'$ such that $u \overset{spec}{\text{—}} v$. The set $C_M(u)$ is a proper hyperplane of M containing l. Therefore there exists a point $z \in M \setminus l$ such that $u \overset{pol}{\text{—}} z$. Pick $y \in l \setminus \{x\}$. For every point w of the line yz either $v \overset{pol}{\text{—}} w$ or $v \overset{spec}{\text{—}} w$ holds.

By Lemma 10, there exists at least one point $w \in yz$ such that $v \overset{pol}{\text{---}} w$. Now we have a contradiction with Lemma 22. $\square$

In view of Proposition 23, we can distinguish two cases:

- The singular rank of $(\mathcal{E}, \mathcal{F})$ equals 2.
- Each point in $\mathcal{E}$ is on precisely 2 maximal singular subspaces.

We shall deal with the former case in the next subsection, and with the latter case in the subsequent (final) subsection of the section.

4.2 Singular rank 2

In this subsection, we assume that $(\mathcal{E}, \mathcal{F})$ has singular rank 2. Recall that each line is in a unique maximal singular subspace. For a line $l \in \mathcal{F}$ we define

$$S(l) = l \cup \{z \in \mathcal{E}_{\leq 0}(l) \mid |\mathcal{E}_{\leq -1}(z) \cap l| = 1\}.$$

Lemma 24 *For $l \in \mathcal{F}$, $(u, v) \in \mathcal{E}_0$, and $x \in \mathcal{E}_{-1}(u, v)$, the following properties hold.*

(i) $S(l) = \mathcal{E}_{\leq 0}(l) \setminus (M(l) \setminus l)$.

(ii) $S(ux) = S(vx)$.

(iii) $S(vx) = \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(u, v))$.

Proof. (i). Suppose $x \in \mathcal{E}_{\leq 0}(l) \setminus M(l)$. By Lemma 16, $C_{M(l)}(x)$ is a proper hyperplane of $M(l)$. As it contains l and $M(l)$ has rank 2, this forces $C_{M(l)}(x) = l$. The unique point of $M(l) \cap \mathcal{E}_{-1}(x)$ belongs to this hyperplane, and hence to l. Therefore, $x \in S(l)$. The other inclusions are trivial.

(ii). By symmetry it is sufficient to show that $S(vx) \subseteq S(ux)$. It is obvious that $vx \subseteq S(ux)$. Pick $w \in S(vx) \setminus (vx \cup ux)$. Observe that $w \notin M(xu)$, for otherwise $C_{M(xu)}(v)$ would contain xuw, whence all of $M(xu)$, contradicting Lemma 16.

Assume that w does not commute with u. Then $u \overset{spec}{\text{---}} w$ by the triangle condition on x, u, w. Set $z = [u, w]$. If $x \overset{coll}{\text{---}} w$, then $x = z$ and we obtain a contradiction to Lemma 22 with x, u, w and some point of xv; therefore $x \overset{pol}{\text{---}} w$. By the filtration around x, we have $x \overset{coll}{\text{---}} z$ and therefore z is on the singular plane $M(ux)$. Furthermore, z is not on the line xu for otherwise each point of $xu \setminus \{z\}$ would be special with w, contradicting $x \overset{pol}{\text{—}} w$. By the filtration around v we have $z \in \mathcal{E}_{\leq 0}(v)$. This means that v commutes with the whole plane $M(ux)$, a contradiction to Lemma 16. We have shown $w \in \mathcal{E}_{\leq 0}(u,x) \setminus M(xu)$, so, by (i), $w \in S(xu)$.

(iii). To see the inclusion $\mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(u,v)) \subseteq S(vx)$ note that $\mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(u,v)) \subseteq \mathcal{E}_{\leq 0}(u,v)$, since $u, v \in \mathcal{E}_{\leq 0}(u,v)$, and that $\mathcal{E}_{\leq 0}(u,v) \subseteq S(vx)$ by Lemmas 16, 20, and (i). To see the reverse inclusion, let $w \in S(vx) \setminus (vx \cup ux) = S(ux) \setminus (vx \cup ux)$ and $z \in \mathcal{E}_{\leq 0}(u,v)$. We need to show $w \in \mathcal{E}_{\leq 0}(z)$. By Lemma 20, $\mathcal{E}_{\leq 0}(u,v) = \mathcal{E}_{\leq 0}(ux \cup vx)$.

If z is not collinear with x then, there are noncollinear $u' \in ux$ and $v' \in vx$ such that $z \in \mathcal{E}_{-1}(u',v')$, and, since $w \in \mathcal{E}_{\leq 0}(u',v')$, Lemma 20 gives $w \in \mathcal{E}_{\leq 0}(z)$.

Arguing as in the previous paragraph, but with w and z interchanged, we find $z \in \mathcal{E}_{\leq 0}(w)$ if w is not collinear with x. Therefore, we are left with the case where both $z \overset{coll}{\text{—}} x$ and $w \overset{coll}{\text{—}} x$. But then we can use Lemma 22 to exclude the possibility of $z \overset{spec}{\text{—}} w$. $\square$

Let $\mathcal{S}$ be the collection of the subspaces of $\mathcal{E}$ of the form $\mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(x,y))$ where $x \overset{pol}{\text{—}} y$. Its members will be called *symplecta*. This terminology is justified by the following consequence of the above lemma.

Corollary 25 *The space $(\mathcal{E}, \mathcal{F})$ is a parapolar space of polar rank 2 and singular rank 2. If $(x,y) \in \mathcal{E}_0$, then $\mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(x,y))$ is the unique symplecton containing x and y. The intersection of two symplecta is either empty or a singleton. The intersection of a symplecton and a singular plane of $(\mathcal{E}, \mathcal{F})$ is either empty or a line.*

Proof. Let $S = \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(x,y))$ where $(x,y) \in \mathcal{E}_0$. Then, by axiom (E), S is a subspace. Therefore S is a gamma space as $(\mathcal{E}, \mathcal{F})$ is. Assume that l is a line of S and $w \in S$. Applying $\mathcal{E}_0$ twicve to both sides of the inclusion $\{u,v\} \subseteq \mathcal{E}_{\leq 0}(u,v)$, we find $S \subseteq \mathcal{E}_{\leq 0}(S)$. Consequently, $w \in \mathcal{E}_{\leq 0}(S) \subseteq \mathcal{E}_{\leq 0}(l)$ and hence, by Lemma 16, either $w \in M(l)$ or there is exactly one point of l collinear with w. Therefore S is a polar space. To see that the polar rank is at least 2, we need to show that S contains at least one line. To this end, recall that, by Proposition 11, $\mathcal{E}_{\leq -1}(x,y)$ is a non-degenerate polar space (in the present case with no lines). Therefore there is a pair $u, v \in \mathcal{E}_{\leq -1}(x,y)$ such that $u \overset{pol}{\text{—}} v$. Repeated applications of Lemma 24 give $S = \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(x,y)) = S(ux) = \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(u,v))$. In particular, $u \in S$ and ux is a line of S. Thus S

is indeed a polar space. Non-degeneracy of S follows from non-degeneracy of $\mathcal{E}_{\leq -1}(x,y)$ which is a subset of S by the preceding argument.

Let l be an arbitrary line of S on x. Then there exists a unique point u of l collinear with y. By non-degeneracy of the polar space $\mathcal{E}_{\leq -1}(x,y)$ there is an element $v \in \mathcal{E}_{\leq -1}(x,y)$ such that $u \overset{pol}{—} v$. Then, as above, repeated applications of Lemma 24 give $S = S(l)$. Let m be a line of S intersecting l in a point u' and let $v' \in S$ be polar to u'. Then we can choose $x' \in l$ and $y' \in m$ collinear with v'. Again from repeated applications of Lemma 24 we infer that $S(m) = S(l) = S$. Finally let m' be a line of S not intersecting l. Then we can find a line m of S connecting a point of l with one of m'. An argument like above shows that $S(m') = S(m)$. We obtain that $S = S(l)$ for an arbitrary line l of S.

Now let x', y' be an arbitrary polar pair of S. As S is a non-degenerate polar space we can embed x', y' into a quadrangle of S. We can use again Lemma 24 to show that $\mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(x',y')) = S(l) = S$ for some line l of S on x' and also $\mathcal{E}_{-1}(x',y') \subseteq S$. As a consequence, S is geodesically closed (in other words, convex).

In view of Lemma 24, any line n in $\mathcal{F}$ is contained in at most one symplecton, namely in $S(n)$. To see that $S(n)$ is indeed a symplecton, observe that by connectedness of $(\mathcal{E}, \mathcal{F})$, there is a line $m \in \mathcal{F}$ intersecting $M(n)$ in a point x of n. Then $M(m)$ intersects $M(n)$ in x. Pick $v \in n \setminus \{x\}$. By Lemma 16, $m' = C_{M(m)}(v)$ is a line containing x and any point u different from x is in $\mathcal{E}_0(v)$. Lemma 24 gives that $S(n) = \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(u,v))$, a symplecton. Thus every line $n \in \mathcal{F}$ is contained in the unique symplecton $S(n)$. As $M(n) \not\subseteq S(n)$, the polar rank must be 2.

The assertions regarding intersections follow from convexity of symplecta. □

Lemma 26 *Let $x, y, z \in \mathcal{E}$ be three pairwise commuting distinct points not lying on a singular plane. Then x, y, and z are contained in a symplecton.*

Proof. By Lemma 24(iii) it suffices to find a line l with $x, y, z \in S(l)$. Assume first that at least two of the three points are collinear, say $x \overset{coll}{—} y$. Then, by Lemma 24(i), $x, y, z \in S(xy)$. If none of the pairs are collinear, choose $u \in \mathcal{E}_{\leq -1}(x,y)$. By Corollary 25, the symplecton containing x and y is $S(xu)$. As $z \in \mathcal{E}_{\leq 0}(x,y)$, Lemma 20 gives $z \in \mathcal{E}_{\leq 0}(xu)$. Since z is not collinear with x, we even have $z \in S(xu)$. □

Lemma 27 *Assume that three distinct symplecta pairwise intersect each other. Then there is a singular plane of $\mathcal{E}$ containing the intersections.*

Proof. By Corollary 25 the intersections are the singletons $\{x\}, \{y\}, \{z\}$. If these points are distinct then, by Lemma 26, they must lie on a singular plane of $\mathcal{E}$. □

Lemma 28 *Let S be a symplecton and $z \in \mathcal{E} \setminus S$.*

(i) Assume that S intersects two distinct symplecta S_1 and S_2 containing z. Then S intersects every symplecton containing z.

(ii) The point z commutes with at least one point of S.

Proof. (i). Let $\{y\} = S \cap S_1$ and $\{x\} = S \cap S_2$. Then x, y, z lie in a singular plane P. Any further symplecton containing z intersects P in a line l on z. As l is a hyperplane of P, it intersects the line $xy = S \cap P$.

(ii). Assume that $S \subseteq \mathcal{E}_{\geq 1}(z)$. First we show that then the set $S \cap \mathcal{E}_1(z)$ contains a polar pair. This is trivial if $S \subseteq \mathcal{E}_1(z)$. Otherwise S contains a point $x \in \mathcal{E}_2(z)$ and there exist two distinct lines l and m of S on x (choose two singular planes of $\mathcal{E}$ through x and take the intersections with S). Furthermore, by (F) there exist $u \in l$ such that $z \overset{\textit{spec}}{\text{———}} u$ and $v \in m$ such that $v \overset{\textit{spec}}{\text{———}} z$. As $u \neq x \neq v$, we have $u \overset{\textit{pol}}{\text{———}} v$.

Assume that (u, v) is a polar pair from $S \cap \mathcal{E}_1(z)$. By filtration around v, we have $[z, u] \in \mathcal{E}_{\leq 1}(v)$. If $[v, [z, u]] = 0$ then $v \in S(u[z, u])$, whence $S(u[z, u]) = \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(u, v)) = S$ and $[z, u]$ is a point of S commuting with z. Otherwise put $w = [v, [z, u]]$. Then $v \overset{\textit{coll}}{\text{———}} w$ (by definition of w), $u \overset{\textit{coll}}{\text{———}} w$ (by filtration around u), and $[z, w] = 0$ (by filtration around z). Thus w is a point of $\mathcal{E}_{\leq -1}(u, v) \subseteq S$ commuting with z. □

We conclude that the space is the Grassmannian of lines in a non-degenerate polar space of rank 3.

Theorem 29 *If $(\mathcal{E}, \mathcal{F})$ is a non-degenerate root filtration space of singular rank 2 such that each line is on a unique maximal singular subspace, then it is a shadow space of type* $(\mathrm{B|C})_{3,2}$.

Proof. By the well-known characterization of shadow spaces of type $\mathrm{C}_{n,1}$ as non-degenerate polar spaces of rank n, it suffices to show that the pair

$(\mathcal{S}, \mathcal{E})$ is a non-degenerate polar space of (polar) rank 3. Here a member of $\mathcal{E}$ is interpreted as a line by identifying it with the collection of symplecta it contains. By Lemma 28, $(\mathcal{S}, \mathcal{E})$ satisfies the polar space property.

Suppose S is a symplecton. Choose $x \in S$ and let T be another symplecton on x. Finally, choose $y \in T \cap \mathcal{E}_0(x)$ and let U be a symplecton on y distinct from T. If $z \in S \cap U$, then, by Lemma 27, z, x, y are in a singular subspace of $(\mathcal{E}, \mathcal{F})$, contradicting $x \overset{pol}{\text{---}} y$. Therefore, U is a point of $(\mathcal{S}, \mathcal{E})$ non-collinear with with S. It follows that $(\mathcal{S}, \mathcal{E})$ is a non-degenerate polar space.

Let M be a maximal singular subspace of $(\mathcal{E}, \mathcal{F})$. So its rank is 2. The image of the morphism $l \mapsto S(l)$ from the dual of M to $\mathcal{S}$ is a maximal singular subspace of $(\mathcal{S}, \mathcal{E})$, so the latter is a non-degenerate polar space of rank 3. $\square$

4.3 Two maximal singulars per point

In this subsection we assume that every line $l \in \mathcal{F}$ is contained in a unique maximal singular subspace $M(l)$ and that every point is contained in exactly two maximal singular subspaces. We further assume that the singular rank is at least 2.

We say that two maximal singular subspaces of $\mathcal{E}$ are *parallel* if either $M_1 = M_2$ or $M_1 \cap M_2 = \emptyset$ and there exist points $x_1 \in M_1$ and $x_2 \in M_2$ such that $x_1 \overset{coll}{\text{---}} x_2$.

Lemma 30 *If M_1 and M_2 are distinct parallel maximal singular subspaces of $\mathcal{E}$ then there exist proper hyperplanes H_i of M_i such that for every point $z_1 \in H_1$ there exists a (unique) point $z_2 \in H_2$ such that $z_1 \overset{coll}{\text{---}} z_2$ and vice versa.*

Proof. Let x_1, x_2 be points of M_1 and M_2, respectively, such that $x_1 \overset{coll}{\text{---}} x_2$. Set $H_1 = C_{M_1}(x_2)$ and $H_2 = C_{M_2}(x_1)$. For $z_1 \in H_1 \setminus \{x_1\}$ we have $x_2 \overset{pol}{\text{---}} z_1$, so by Lemma 16 there exists a (unique) point $z_2 \in M_2 \cap \mathcal{E}_{-1}(z_1)$. As $z_1, x_2 \in \mathcal{E}_{\leq -1}(x_1, z_2)$, the pair x_1, z_2 must be polar, whence $z_2 \in C_{M_2}(x_1) = H_2$. $\square$

Lemma 31 *Parallellity is a transitive relation.*

Proof. Let M_1, M_2, M_3 be distinct maximal singular subspaces such that M_1 is parallel to M_2 and M_2 is parallel to M_3. By the previous lemma and rank considerations there exist points $x_i \in M_i$ such that $x_1 \overset{coll}{\text{——}} x_2$ and $x_2 \overset{coll}{\text{——}} x_3$. As $M_2 \cap M_1 = \emptyset$ the line x_1x_2 is not contained in M_2. The same holds for the line x_2x_3. Therefore the points x_1 and x_3 are contained in the other maximal singular subspace through x_2. In particular, x_1 and x_3 are either equal or collinear. Equality cannot happen: then M_1 and M_3 would be two distinct maximal singular subspaces on x not containing the line x_1x_2. Thus we have $x_1 \overset{coll}{\text{——}} x_3$. Assume that $M_1 \cap M_3$ is not empty. Then it must be a singleton $z \notin \{x_1, x_3\}$. Then x_1, x_3, z are pairwise collinear, so they are contained in a maximal singular subspace M. As M contains the lines zx_1 and zx_3, it must be equal to both M_1 and M_3, contrary to assumption. Thus M_1 and M_3 are disjoint, and therefore parallel. $\Box$

Lemma 32 *Let M be a maximal singular subspace and $x \in \mathcal{E} \setminus M$. Then there is a singular subspace parallel to M through x.*

Proof. Assume first that x is in hyperbolic relation with a point of M. Then, by Lemma 16, none of the maximal singular subspaces through x intersects M and there is a point $x_1 \in M \cap \mathcal{E}_1(x)$. Set $x_2 = [x, x_1]$ and let N be the maximal singular subspace containing the line xx_2. Then N is as required.

Next assume that no point of M is hyperbolic with x. Again by Lemma 16 there is a point x_1 of M collinear with x. Then the maximal singular subspace through x not containing the line x_1x is as required. $\Box$

We have shown that there are exactly two classes of parallel maximal singular subspaces: Fix a point $y \in \mathcal{E}$, let M_1 and M_2 be the maximal singular subspaces on y, and let $\mathcal{M}_1$ be the class of M_1 and $\mathcal{M}_2$ the class of M_2. Then for every point $x \in \mathcal{E}$, one of the maximal singulars containing x belongs to $\mathcal{M}_1$ while the other belongs to $\mathcal{M}_2$.

Proposition 33 *Let $i \in \{1, 2\}$, and let M and N be distinct members of $\mathcal{M}_i$. For collinear points $x_1 \in M$ and $x_2 \in N$, define the line MN as the set of the maximal singular subspaces in $\mathcal{M}_i$ intersecting the line x_1x_2. This definition is independent of the choice of x_1 and x_2. Moreover, with this definition of lines, $\mathcal{M}_i$ is a projective space.*

Proof. Suppose z_1, z_2 are also collinear with $z_1 \in M$ and $z_2 \in N$. Assume that K is a member of $\mathcal{M}_i$ distinct from M and N and intersecting the line x_1x_2 in the point x_3. Then, as x_1, z_2 is a polar pair, by Lemma 21 there is a point z_3 on the line $z_1 z_2$ collinear with x_3. Also, z_3 must be polar to x_1, so the maximal singular subspace containing the line $x_3 z_3$ must be K. We conclude that the definition of MN does not depend on the choice of x_1, x_2.

Let X_1, X_2, X_3 be three members of $\mathcal{M}_i$ not all on one line. As in the proof of transitivity, there exist $x_i \in X_i$ $(i = 1, 2, 3)$ such that $x_1 \overset{coll}{\text{—}} x_2$ and $x_2 \overset{coll}{\text{—}} x_3$. Again, as the lines $x_1 x_2$ and $x_2 x_3$ do not lie in X_2, they must be contained in the other maximal singular subspace, Y say, through x_2 (in particular, $x_1 \overset{coll}{\text{—}} x_3$ as well). Now, by taking intersections with Y, we obtain an isomorphism between the plane of maximal singular subspaces spanned by X_1, X_2, X_3 and the projective plane in Y spanned by x_1, x_2, x_3. $\Box$

Lemma 34 *Let $i \in \{1, 2\}$, and let M be in $\mathcal{M}_i$. Then the set of the maximal singular subspaces intersecting M form a proper hyperplane of $\mathcal{M}_{3-i}$. The resulting map from $\mathcal{M}_i$ to the dual of $\mathcal{M}_{3-i}$ is an injective morphism of linear spaces.*

Proof. If M intersects the maximal singular subspaces X_1 and X_2 in the point x_1 and x_2, respectively, then $x_1 x_2$ is a line of M and the line $X_1 X_2$ consists of the maximal singular subspaces in $\mathcal{M}_{3-i}$ through a point of $x_1 x_2$. This shows that the set in question forms a subspace. Let $x \in \mathcal{E}$ be a point which is hyperbolic to some point of M. Then x cannot be collinear with any point of M, so the maximal singular subspace on x in $\mathcal{M}_{3-i}$ does not intersect M. This shows that the subspace is proper.

To see that it is indeed a hyperplane of $\mathcal{M}_{3-i}$, assume that $X_1 X_2$ is a line in $\mathcal{M}_{3-i}$. Again, this line is given by a line $x_1 x_2$ where $x_i \in X_i$ and $x_1 \overset{coll}{\text{—}} x_2$. Consider the maximal singular subspace Y containing the line $x_1 x_2$. As Y intersects X_1, it must be in $\mathcal{M}_i$, so, by Lemma 30, the points of Y collinear with some point of M form a proper hyperplane of M. In particular, there is a point x_3 on the line $x_1 x_2$ which is collinear with a point x of M. Now M intersects the maximal singular subspace $M(x_3 x)$ which lies on the line $X_1 X_2$.

The final assertion follows easily from the previous ones. $\Box$

As a consequence, we have the following result. Recall the definition of $\mathcal{E}(\mathbb{P}, \mathbb{H})$ from the beginning of Section 4.

Theorem 35 *Suppose that $(\mathcal{E}, \mathcal{F})$ is a non-degenerate root filtration space of singular rank at least 2 such that each line is on a unique maximal singular subspace and such that there are precisely two maximal singular subspaces* *on some point. Then the collection of maximal singular subspaces can be partitioned into two sets $\mathcal{M}_1$, $\mathcal{M}_2$, each of which carries the structure of a projective space whose lines are the members of the class containing exactly one point of a line in $\mathcal{F}$. Moreover, the space $\mathcal{M}_2$ can be viewed as a subspace of the dual of $\mathcal{M}_1$ annihilating $\mathcal{M}_1$, and, with this identification, $(\mathcal{E}, \mathcal{F})$ is isomorphic to $\mathcal{E}(\mathcal{M}_1, \mathcal{M}_2)$.*

In particular, if the singular rank of $(\mathcal{E}, \mathcal{F})$ is finite, say $n-1$ ($n \geq 3$), and all lines are thick, then $(\mathcal{E}, \mathcal{F})$ is a shadow space of type $\mathrm{A}_{n,\{1,n\}}$.

Proof. After the identification of $\mathcal{M}_2$ with a subspace of the dual of $\mathcal{M}_2$, the pair of projective spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy the requirements for $\mathcal{E}(\mathcal{M}_1, \mathcal{M}_2)$ to be well defined. Indeed, the fact that for each member of $\mathcal{M}_1$ there is a disjoint member of $\mathcal{M}_2$ implies that $\mathcal{M}_2$ annihilates $\mathcal{M}_1$.

Since each point of $\mathcal{E}$ lies on a unique member of each $\mathcal{M}_i$ ($i = 1, 2$), there is a well-defined map $\mathcal{E} \to \mathcal{M}_1 \times \mathcal{M}_2$, whose image can be identified with the set of incident pairs of $\mathcal{M}_1 \times \mathcal{M}_2$, that is, the point set of $\mathcal{E}(\mathcal{M}_1, \mathcal{M}_2)$. It is straightforward to verify that this map is an isomorphism of spaces $(\mathcal{E}, \mathcal{F}) \to \mathcal{E}(\mathcal{M}_1, \mathcal{M}_2)$.

If $(\mathcal{E}, \mathcal{F})$ has finite rank, then $\mathcal{M}_2$ must be the dual of $\mathcal{M}_1$. If, moreover, $\mathcal{M}_1$ has thick lines only, there is a natural number $n \geq 3$ and a division ring $\mathbb{D}$ such that $\mathcal{M}_1 \cong \mathbb{P}(\mathbb{D}^{n+1})$. It follows that $(\mathcal{E}, \mathcal{F})$ is isomorphic to the shadow space of the building of type A_n on $\mathbb{P}(\mathbb{D}^{n+1})$, whence a shadow space of type $\mathrm{A}_{n,\{1,n\}}$. $\square$

Combining Corollary 18, Proposition 23 and Theorems 29 and 35, we obtain a proof of Theorem 15. This result, together with Theorems 29 and 35, establishes Theorem 1.

5 Root shadow spaces

In this section, we prove the following converse of Theorem 1.

Theorem 36 *Suppose that Y_n is an irreducible Dynkin diagram. Then the root shadow space of a building of type Y_n is either a non-degenerate polar space or a non-degenerate root filtration space.*

In Proposition 39, we show that root shadow spaces are either non-degenerate polar spaces or non-degenerate root parapolar spaces and in Proposition 41 we establish that non-degenerate root parapolar spaces are non-degenerate root filtration spaces.

We shall approach buildings 'locally', that is, by means of chamber systems as initiated in [12]. We refer the reader to [10] for terminology and notation and recall that we use the Coxeter diagram labeling of [2]. For the remainder of this section, let $\mathcal{C}$ be a chamber system over $I = \{1, \dots, n\}$ which is a building of Dynkin type Y_n. We let $J \subseteq I$ be the set of non-root nodes. For c a chamber of $\mathcal{C}$, we denote by cJ^* the J-residue of c and, for $X \subset \mathcal{C}$, by X/J the set of all J-residues intersecting X. We write W for the Coxeter group of $\mathcal{C}$ and W_J for its parabolic subgroup with respect to J. There is a W-distance function $\delta : \mathcal{C} \times \mathcal{C} \to W$. It induces a $W_J \backslash W / W_J$-distance $\delta_J : \mathcal{C}/J \times \mathcal{C}/J \to W_J \backslash W / W_J$ on the J-residues given by $\delta_J(cJ^*, dJ^*) = W_J \delta(c, d) W_J$.

We let $(\mathcal{E}, \mathcal{F})$ be the root shadow of $\mathcal{C}$ with respect to Y_n. Thus, $\mathcal{E} = \mathcal{C}/J$ is the set of J-residues of $\mathcal{C}$ and the members of $\mathcal{F}$ are of the form $c\{r\}^*/J$ for some $r \in I \setminus J$.

Proposition 37 *Let $\mathcal{C}$ be a building of rank at least 3 and of spherical Dynkin type Y_n. Denote by W the Coxeter group of type Y_n and by $(\mathcal{E}, \mathcal{F})$ the root shadow space of $\mathcal{C}$ with respect to Y_n.*

(i) $(\mathcal{E}, \mathcal{F})$ is a parapolar space. Its symplecta are the collections cT^*/J, where c is a chamber of $\mathcal{C}$ and $T \subseteq I$ has an induced subdiagram Z_m of Y_n equal to B_m, C_m, or D_m for some $m \leq n$, such that the intersection with J has diagram Z_{m-1}.

(ii) If $w \in W$ is left and right J-reduced, and c is a chamber of $\mathcal{C}$, then the set $S_{\leq w}(cJ^*) := \{dJ^* \mid d \in \mathcal{C}, \delta(c, d) \leq w\}$, where $\leq$ stands for the Bruhat order, is a subspace of $(\mathcal{E}, \mathcal{F})$.

Proof. Let J be the complement in I of the root nodes. Then $\mathcal{E} = \mathcal{C}/J$ is the set of J-residues in $\mathcal{C}$.

(i). By Lemma 10.6.4 of [3], $(\mathcal{E}, \mathcal{F})$ is a partial linear space. By arguments similar to those for Theorem 10.6.3 of [3], it follows that $(\mathcal{E}, \mathcal{F})$ is a gamma space, and by arguments similar to those for Theorem 10.2.6, Proposition 10.6.8, and the subsequent remarks of [3], it follows that $(\mathcal{E}, \mathcal{F})$ is a parapolar space.

(ii). Suppose $d\{r\}^*/J$ with $r \in I \setminus J$ and $d \in \mathcal{C}$ is a line with two points, say dJ^* and bJ^* in $S_{\leq w}(cJ^*)$. The set $\{\delta(c, e) \mid e \in d\{r\}^*/J\}$ is of the form

$\{v, vr\}W_J$ for some $v \in W$. Without loss of generality, we can choose v such that $v \leq vr$ (for, v and vr can be interchanged). By the Gate Property (see [10], Theorem (2.9)), there is a unique chamber $a \in d\{r\}^*$ with $\delta(c, a) = v$. At least one of bJ^* and dJ^* is distinct from aJ^*, say bJ^* is. Then $\delta(c, b) = vr$. In particular, $vr \leq w$, and hence also $v \leq w$, so each point of $d\{r\}^*/J$ belongs to $S_{\leq w}(cJ^*)$. $\square$

By Φ we denote the root system of type Y_n and by α_0 the highest (long) root of Φ. Then W is the Weyl group of Φ and W_J is the stabilizer in W of α_0. Let ϕ be the W-equivariant embedding $W/W_J \to \Phi$ sending W_J to α_0. Its image is the set Ψ of long roots of Φ, which is itself a root system, and the $W_J\backslash W/W_J$-distance of the pair (uW_J, vW_J) of cosets (that is, $W_Jv^{-1}uW_J$) uniquely determines the inner product $(\phi(uW_J), \phi(vW_J))$. To be more precise, viewing W/W_J as the set of points of the root shadow space of the thin building on W, the value of the inner product equals $2, 1, 0, -1, -2$ according as the pair (uW_J, vW_J) of points are equal, collinear, polar, special, or without common neighbors, respectively. Observe that the correspondence between double cosets and inner products need not be bijective: In the case $Y_n = \mathrm{A}_n$ ($n \geq 2$), the pairs $r_1W_J, W_J)$ and $r_nW_J, W_J)$, where r_1 and r_n are distinct fundamental reflections corresponding to extremal nodes, represent different double cosets in the group although both are pairs of collinear points in the root shadow geometry. See [3], Chapter 10, for further details.

Lemma 38 *Suppose $Y_n \neq \mathrm{C}_n$. If α, β, $\gamma \in \Psi$ satisfy $(\alpha, \beta) = 0$ and $(\alpha, \gamma) = 1$ but $(\beta, \gamma) \neq 1$, then there is $\zeta \in \Psi$ with $(\alpha, \zeta) = (\beta, \zeta) = (\gamma, \zeta) = 1$.*

Proof. If $(\beta, \gamma) = -1$, then the choice $\zeta = r_\beta\gamma$ suffices. In view of the assumption $(\beta, \gamma) \neq 1$ and $\beta \neq \pm\gamma$, we may therefore assume $(\beta, \gamma) = 0$.

Since $Y_n \neq \mathrm{C}_n$ and Φ is irreducible, Ψ is also irreducible, and so there is a root $\epsilon \in \Psi$ with $(\alpha, \epsilon) = (\beta, \epsilon) = 1$. If $(\gamma, \epsilon) = 1$ the choice $\zeta = \epsilon$ suffices. Clearly, $(\epsilon, \gamma) \neq \pm 2$, so it remains to consider the cases $(\epsilon, \gamma) = 0, -1$.

The vector $\eta := \alpha + \beta - \epsilon = r_\alpha r_\beta(-\epsilon)$ is also a root in Ψ with $(\alpha, \eta) = (\beta, \eta) = 1$. In particular, if $(\eta, \gamma) = 1$, the choice $\zeta = \eta$ suffices. But $(\eta, \gamma) = 1 + (\beta, \gamma) - (\epsilon, \gamma)$, so we have dealt with the case where $(\epsilon, \gamma) = (\beta, \gamma) = 0$. The case where $(\epsilon, \gamma) = -1$ remains. But then $(\eta, \gamma) = 2$, so $\gamma = \eta$, a contradiction with the assumption $(\beta, \gamma) \neq 1$. $\square$

Proposition 39 *Let $(\mathcal{E}, \mathcal{F})$ be the root shadow space of a spherical building of irreducible Dynkin type Y_n with $n \geq 3$. If $Y_n = \mathrm{C}_n$, then $(\mathcal{E}, \mathcal{F})$ is a* *non-degenerate polar space. If $Y_n \neq C_n$, then $(\mathcal{E}, \mathcal{F})$ is a non-degenerate root parapolar space.*

Proof. If $Y_n = C_n$, then $(\mathcal{E}, \mathcal{F})$ is a root shadow space of type $C_{n,1}$, which is well known to be a non-degenerate polar space. Therefore, we assume $Y_n \neq C_n$. By Proposition 37, $(\mathcal{E}, \mathcal{F})$ is a parapolar space, so we only need to verify conditions (a), (b), (c) of the definition of a non-degenerate root parapolar space.

Let c be a chamber of $\mathcal{C}$ and A an apartment of $\mathcal{C}$ containing c. Composition of the canonical bijection $A \to \{c\} \times A$ with the map δ gives a W-distance preserving bijection $A \to W$, where the W-distance between x and y in W is by definition $x^{-1}y$. This induces a W-equivariant isomorphism $\rho : A/J \to W/W_J$ sending cJ^* to W_J. The composition of ρ and ϕ is a W-equivariant isomorphism $A/J \to \Psi$ which we denote by σ. Let $x, y \in A/J$ and write $\alpha = \sigma(x)$, $\beta = \sigma(y)$. Then, by [3], Theorem 10.2.6 and Proposition 10.6.8, α, $\beta \in \Psi$ satisfy $(\alpha, \beta) = 2$ if and only if $x = y$; $(\alpha, \beta) = 1$ if and only if x and y are collinear (recall this implies distinct); $(\alpha, \beta) = 0$ if and only if x and y are not collinear but contained in a symplecton; $(\alpha, \beta) = -1$ if and only if x and y are not collinear but have a unique common neighbor; and $(\alpha, \beta) = -2$ otherwise.

(a). Suppose that S is a symplecton of $(\mathcal{E}, \mathcal{F})$ and x is a point outside S collinear with a point y of S. Then there are chambers c and d of $\mathcal{C}$ such that $y = cJ^* = dJ^*$, $S = cT^*/J$, where T is the type set of S, and $xy = d\{r\}^*/J$ for some $r \in I \setminus J$. Embedding c and d in an apartment A and applying the map σ defined above, we find roots $\alpha = \sigma(y)$ and $\beta = \sigma(z)$ corresponding to a point z of S non-collinear with y with $(\alpha, \beta) = 0$, and a root $\gamma = \sigma(v)$ with $(\alpha, \gamma) = 1$ representing a point v of $xy \setminus \{y\}$. Since $x \notin S$, we have $v \notin S$ (cf. Proposition 37(i)) and so $(\gamma, \beta) \neq 1$ (for otherwise, v would be collinear with both y and z, whence in S). By Lemma 38, there is a root $\zeta \in \Psi$ with $(\alpha, \zeta) = (\beta, \zeta) = (\gamma, \zeta) = 1$. Consequently, $u := \sigma^{-1}(\zeta)$ is a point collinear with v, y, and z. In particular, $u \in S$, so the entire line yu lies in S and, by the gamma space property of the parapolar space $(\mathcal{E}, \mathcal{F})$, the line yu consists of points collinear with x. This proves (a).

(b). Suppose that π is a singular plane meeting the line l at a point x. Then we can find chambers c and d such that $x = cJ^* = dJ^*$, $\pi = cK^*/J$ and $l = dL^*/J$ for certain $K, L \subseteq I$ such that K is the type of a plane and L is the type of a line. Fix an apartment A containing c and d, and consider images of its elements under σ. The image $\sigma(\pi \cap (A/J))$ consists of three long roots, α, β, γ, one of which, say γ, coincides with $\sigma(x)$. Moreover $\sigma(l \cap (A/J))$ consists of two long roots of Φ, one of which is γ; call the other ζ and write $v = \sigma^{-1}(\gamma)$ for the corresponding point. Set $a = (\alpha, \zeta)$ and $b = (\beta, \zeta)$. We assume that $\langle l, \pi \rangle$ is not singular; this means that a and b are distinct from 2 and that at least one of a, b is in $\{0, -1\}$. The 4×4 Gram matrix on α, β, γ, ζ is

$$\begin{pmatrix} 2 & 1 & 1 & a \\ 1 & 2 & 1 & b \\ 1 & 1 & 2 & 1 \\ a & b & 1 & 2 \end{pmatrix},$$

with determinant $5 + 2b - 3b^2 + 2a + 2ba - 3a^2$. This determinant is negative if a or b is in $\{-1, -2\}$, contradicting that a Gram matrix is positive semi-definite. In particular $a, b \in \{0, 1\}$ and at least one of α, β has inner product 0 with ζ. This implies that there is a symplecton on v and some point of π distinct from x. By Proposition 37(ii), the union of all symplecta on v is a subspace of $(\mathcal{E}, \mathcal{F})$. Since, by definition, π is also a subspace, it follows that the set of all points w of π for which there is a symplecton containing w and v is either a line on x or coincides with π. This establishes (b).

(c). Let x and y be distinct collinear points. This means that there are chambers $c \in x$ and $d \in y$ such that $d \in c\{r\}^*$. That is, c and d belong to the same $\{r\}$-panel, $x = cJ^*$ and $y = dJ^*$. By the classical axioms of building theory, there exists an apartment A containing c and d. Note that x and y both meet A in some chamber and hence they meet A. Using σ to identify J-residues meeting A with roots, we therefore find long roots $\alpha = \sigma(x)$ and $\beta = \sigma(y)$. Since x and y are collinear, we have $(\alpha, \beta) = 1$. Now $\zeta = \alpha - \beta$ is a long root with $(\zeta, \alpha) = 1$ and $(\zeta, \beta) = -1$. Hence $z = \sigma^{-1}(\zeta)$ is a point on A. From $(\zeta, \alpha) = 1$ and $(\zeta, \beta) = -1$ it follows that, inside A, the point z is not collinear to y whereas x is the unique point collinear to z and y. By Theorem 10.6.8 of [3], this holds in $(\mathcal{E}, \mathcal{F})$ as well. □

In order to derive that a root parapolar space is a root filtration space, we define the following relations $\mathcal{E}_i$ $(i = -2, -1, 0, 1, 2)$ on the point set of a root parapolar space $(\mathcal{E}, \mathcal{F})$. The relation $\mathcal{E}_{-2}$ stands for the identity and $\mathcal{E}_{-1}$ for collinearity for distinct points; $\mathcal{E}_0$ stands for being contained in a symplecton but not collinear; the relation $\mathcal{E}_1$ stands for being at distance 2 with a unique common neighbor, and $\mathcal{E}_2 = (\mathcal{E} \times \mathcal{E}) \setminus \mathcal{E}_{\leq 1}$. Observe that, by definition of parapolar space, pairs of points in $\mathcal{E}_2$ have mutual distance at least 3. The notation and terminology for these relations will be as for root filtration spaces. In particular, we also define the map $[\cdot,\cdot] : \mathcal{E}_1 \to \mathcal{E}$ by assigning to a special pair its unique common neighbor in the collinearity graph.

Recall from Lemma 2 that a pentagon in a graph is understood to be an induced pentagon.

Lemma 40 *Any non-degenerate root parapolar space $(\mathcal{E}, \mathcal{F})$ satisfies the following properties.*

(i) Each line is contained in a singular plane.

(ii) If x, y, z are points with $(x, y) \in \mathcal{E}_{-1}$ and $(y, z) \in \mathcal{E}_0$, then there is a point collinear with x, y, and z.

(iii) All non-collinear pairs of points of a pentagon are polar.

Proof. (i). Let $l \in \mathcal{F}$. By (c) there is a line n meeting l at a point x and such that each point of n distinct from x is special with each point of l distinct from x. Let T be a symplecton on n and y a point of l distinct from x. Then y does not belong to T, and x is collinear with y. By (a), there is a line m on x in T collinear with y. Now $\langle l, m\rangle$ is a singular plane on l.

(ii). Let S be the symplecton containing both y and z. Then x is collinear with the point y of S, whence, by (a), with a line m of S containing y. Since S is a polar space, there is a point on m collinear with z. This point is as required.

(iii). Let a, b, c, d, e be a pentagon. We first show that at least one non-collinear pair must be polar. Suppose the contrary, that is, all non-collinear pairs are special. By (i), there is a plane π on de. By (b) there is a point y on π and a symplecton S on y and c. By (ii), there is a point z collinear with b, c, and y. By (b) applied to the point a and the plane bcz, there is a point f on cz and a symplecton T containing f and a. Both f and the line yd lie in the polar space S, and so there is a point g on yd collinear with f. Now consider the pentagon P consisting of a, e, g, f, b. Observe that it is a pentagon indeed (e.g., if $a \overset{coll}{\text{---}} g$, then $a \overset{pol}{\text{---}} d$, a contradiction). In P the pair a, f is polar.

Suppose $e \overset{spec}{\text{---}} f$. Then $g = [e, f]$. Moreover, a is collinear with e and polar with f, so, by (a) applied to the symplecton T and the point e, the point

a must be collinear with g, contradicting that P is a pentagon. Therefore, $e \overset{pol}{—} f$. As b is collinear with f, it follows again from (a) that f is collinear with $[b, e] = a$. This contradicts again that P is a pentagon.

The conclusion is that there are no pentagons all of whose non-collinear pairs are special.

Suppose now that a, b, c, d, e are a pentagon with $a \overset{pol}{—} c$. Then the symplecton S on a and c contains b as well as a point f collinear with a, c and d. In particular, f, being collinear to c, is distinct from e, so the pair a, d, having two distinct neighbors (viz. e and f), must be polar. Going round the pentagon, we see that all non-collinear pairs are in fact polar. $\square$

Proposition 41 *Every non-degenerate root parapolar space is a nondegenerate root filtration space.*

Proof. Let $(\mathcal{E}, \mathcal{F})$ be a non-degenerate root parapolar space. As it is a parapolar space, it is a partial linear space. We verify the axioms (A)–(H) for $(\mathcal{E}, \mathcal{F})$ to be a non-degenerate root filtration space. (A) and (B) follow directly from the definitions of $\mathcal{E}_{-2}$ and $\mathcal{E}_{-1}$.

(C). Let $x, u, v \in \mathcal{E}$ and $i, j \in \{-2, \ldots, 2\}$. Suppose $u \overset{spec}{—} v$ and assume $u \in \mathcal{E}_i(x)$ and $v \in \mathcal{E}_j(x)$. We need to show that $[u, v] \in \mathcal{E}_{\leq i+j}(x)$. Without loss of generality, we may take $-2 \leq i \leq j \leq 2$ with $i + j < 2$.

Suppose $i = -2$. Then $u = x$ and $j = 1$ so $[u, v] = [x, v]$ is collinear with x, as required.

Next, suppose $i = -1$. If $j = -1$, then $x = [u, v]$ by the definition of special pair, so there is nothing to prove. If $j = 0$, then by Lemma 40(ii) there is a point collinear with x, u and v, which must be $[u, v]$, and so $[u, v] \in \mathcal{E}_{\leq -1}(x)$, as required. If $j = 1$, then due to Lemma 40(iii) the 5-circuit x, u, $[u, v]$, v, $[x, v]$ cannot be a pentagon. Since no other pair from the five points can be collinear, $[u, v]$ and $[x, v]$ must be collinear. If they are equal, then $[u, v]$ is collinear to x, and there is nothing left to prove. Otherwise, there are at least two common neighbors to x and $[u, v]$, so $[u, v] \in \mathcal{E}_{\leq 0}(x)$, as required. If $j = 2$, then $[u, v]$, being a neighbor of u, is at distance at most 2 to x, whence in $\mathcal{E}_{\leq 1}(x)$.

Now suppose $i = 0$. If $j = 0$, take u_1 to be a point collinear with x, u and $[u, v]$ (which exists by Lemma 40(ii)) and, likewise, v_1 to be a point collinear with x, v and $[u, v]$. If u_1 and v_1 are distinct, then $[u, v]$ must be in $\mathcal{E}_{\leq 0}(x)$, so assume they coincide. But then u_1 is a common neighbor of u and v and so coincides with $[u,v]$, whence $[u,v] = u_1 \in \mathcal{E}_{-1}(x)$. This shows that $[u,v]$ belongs to $\mathcal{E}_{\leq 0}(x)$.

If $j = 1$, then by Lemma 40(ii) there is a point collinear with x, u, and $[u,v]$. Consequently, $[u,v]$ and x are at distance at most 2, whence $[u,v] \in \mathcal{E}_{\leq 1}(x)$.

This ends the verification of the filtration axiom (C).

(D). Suppose $x \overset{hyp}{\text{—}} y$. If a were to belong to $\mathcal{E}_{\leq -1}(x) \cap \mathcal{E}_{\leq 0}(y)$, then, by Lemma 40(iii), there is a point b collinear with x, y (and a), so $y \in \mathcal{E}_{\leq 1}(x)$, contradicting the hypothesis $x \overset{hyp}{\text{—}} y$. Hence $\mathcal{E}_{\leq -1}(x) \cap \mathcal{E}_{\leq 0}(y)$ is empty.

(E). Let $x \in \mathcal{E}$. Since $(\mathcal{E}, \mathcal{F})$ is a gamma space, $\mathcal{E}_{\leq -1}(x)$ is a subspace.

Suppose that $y, z \in \mathcal{E}_{\leq 0}(x)$ are distinct and collinear. In view of the previous case, we may assume that at least one of these, say z, is not collinear with x. If some point of yz is collinear with x, then yz is contained in the symplecton spanned by x and z, and so lies in $\mathcal{E}_{\leq 0}(x)$. Otherwise, by Lemma 40(ii), there is a point u in $\mathcal{E}$ such that u is collinear with x, y, z. Now applying (b) to the line xu and the plane uyz we conclude that all points of yz are polar with x. Hence $\mathcal{E}_{\leq 0}(x)$ is a subspace of $(\mathcal{E}, \mathcal{F})$.

(F). Let $x \in \mathcal{E}$. We show that $\mathcal{E}_{\leq 1}(x)$ is a geometric hyperplane of $(\mathcal{E}, \mathcal{F})$. Suppose that S is a symplecton. We claim that there is a symplecton T containing x and meeting S nontrivially. To see this, we argue by length of a path in the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ (which is connected as $(\mathcal{E}, \mathcal{F})$ is a parapolar space) from x to S. If this length is 1, then it follows from the existence of a symplecton on a line joining x to a point of S (a parapolar space property). If the length of this path is 2, let y be a point collinear to x and to a point of S. By (a) there is a line of S collinear with y, and by (b), at least one point, say z on this line is polar with x. Now the symplecton on x and z is as required. Suppose therefore, that the distance of x to S is at least 3. By induction on the length, there is a symplecton U containing a point v collinear with x and point w of S. By Lemma 40(ii) there is a point c collinear with x, v and w. But then x has distance at most 2 to S, a contradiction, so we have established the claim.

Suppose that $l \in \mathcal{F}$ has two points in $\mathcal{E}_2(x)$. Take a symplecton S containing l. By the above claim and property (D), there is a point, say p, of S polar with x. By the polar space property of S, there is a point q on l collinear with p. By Lemma 40(ii), there is a point a collinear with p, q and x. But then x has distance at most 2 to the point q of l, so l has a point in $\mathcal{E}_{\leq 1}(x)$.

In order to establish that $\mathcal{E}_{\leq 1}(x)$ is a geometric hyperplane, it remains to show that $\mathcal{E}_{\leq 1}(x)$ is a subspace. Suppose that y, z are distinct collinear points of $\mathcal{E}_{\leq 1}(x)$. In view of (E), we may assume that at least one of them, say z, is special to x. If y and x lie in a symplecton S, then by Lemma 40(ii) there is a point u collinear with x, y, z, in which case all points of yz are collinear to u whence in $\mathcal{E}_{\leq 1}(x)$. Therefore, we may assume that both y and z are special to x. But then, by Lemma 40(iii), the 5-circuit x, $[x, y]$, y, z, $[z, x]$ is not a pentagon. This implies that $[x, y]$ and $[x, z]$ are collinear (observe that y cannot be collinear with $[x, z]$ because of $x \overset{spec}{\text{---}} y$). If $[x, y] = [x, z]$, then $\langle y, z, [x, y]\rangle$ is a singular plane, so (b) gives that there is at least one element on yz contained in a symplecton with x, and we can finish as above. Otherwise, each point of yz is collinear with a point of the line joining $[x, y]$ and $[x, z]$ (as all these points belong to a single symplecton), and so has distance at most 2 to x, whence belongs to $\mathcal{E}_{\leq 1}(x)$. This proves that $\mathcal{E}_{\leq 1}(x)$ is a subspace, whence (F).

(G) follows from (c) and Lemma 3 (recall that this lemma was proved from axioms (A)-(F) only), while (H) is a property of parapolar spaces and so of $(\mathcal{E}, \mathcal{F})$. □

From Propositions 39 and 41 we conclude that root shadow spaces of buildings of rank at least 3 are either non-degenerate polar spaces or non-degenerate root filtration spaces. Since root spaces of buildings of rank 1 are non-degenerate polar spaces of rank 1 and root spaces of buildings of irreducible Dynkin type of rank 2 are either generalized hexagons (cases A_2 and G_2) or generalized quadrangles (cases, B_2 and C_2), this proves Theorem 36.

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