Leonhard Euler's derivation of the water-hammer equations in 1775

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The present paper describes the first known appearance of the water-hammer equations in the archival history. The equations were derived in an article by Leonhard Euler that may have been submitted for a prize competition in 1742, was presented to the Petersburg Academy in 1775, was partially published in 1862, and was finally completed with a newly discovered missing fragment in 1979. Study of this historical work reveals a significant scientific and mathematical breakthrough that underpins much of late 19th and early 20th century developments in transient hydraulics. Here, the history of this remarkable achievement is traced, and the work is placed in historical context as it relates to the theory of fluid transients today.

1 HISTORICAL BACKGROUND

1.1 Biographical overview of Euler's life

Leonhard Euler was born in Basel, Switzerland, in 1707. He joined the university of Basel as a student aged thirteen in the Faculty of Arts. At the time, the famous John Bernoulli, one of the leading mathematicians in Europe, was a Professor of Mathematics in the university. He took Euler under his tutelage and enabled Euler's rapid growth in mathematical abilities. Euler's development under John Bernoulli had a formative and lasting influence on his future career in science and mathematics.

Euler published his first paper in mathematics at the age of nineteen. He showed his exceptional talent for mathematics the next year, 1727, when he sent his first submission for the prestigious Paris prize competition. A Paris prize was widely regarded in those days as the highest scientific accolade, and Euler would go on to win it twelve times, a number never matched by anyone else.

In 1727, Euler moved to Petersburg in Russia, where he would remain active for fourteen years. During this time, in 1738, he lost sight in one of his eyes after a severe bout of fever. He left for Berlin, Germany (Prussia at the time) in 1741, and worked there for 25 years. Eventually, Jean d'Alembert, a mathematician and renowned free-thinker (or *Philosophe*) helped arrange for Joseph-Louis Lagrange, a rising star in mathematics, to join the Berlin Academy as Euler's successor, which opened the path for Euler to return to Petersburg in 1766, aged 59. He stayed there for the rest of his life and died in 1783 at the age of 76.

Around 1768, Euler underwent a botched surgery for cataract, which led to loss of sight in his other eye and left him practically blind. And yet, at 61 years old, Euler's scientific productivity intensified: he published about 400 papers (almost half of his lifetime output) in this state of near blindness. He worked in his study, where he had a table with a slate top on which he could still write a few large equations. Discussing around this table with his assistants, Euler would present a problem he was mulling and suggest a way of solving it. Usually by the next day, an assistant would deliver a rough sketch for a paper based on the previous day's discussions. This would then blossom into a complete paper in about a week. In the year 1775 for example, Euler published more than 50 papers ranging between 10 to 50 pages and concerning a diverse variety of separate problems.

Euler's prodigious exploits have fuelled the attribution of a number of legendary feats of memory and mental arithmetic to him that can be hard to verify. For instance, he is said to have been able to recite the *Aeneid* from start to finish, even recalling the specific sentences located at the top and bottom of each page in the edition that he read as a young boy. For more details on this incredible scientist's life, see the brief biography by Truesdell (1) or the more extensive full-scale biography by Calinger (2).

Figure 1. Oil portrait of Leonhard Euler in 1756, aged 49, by Emanuel Handmann (courtesy of Wikimedia Commons)

1.2 Euler's legacy in science

There is scarcely a topic in classical mechanics or mathematics to which Euler has not made an important contribution. Today's student of mechanical engineering for example can expect to encounter, among others:

- Euler's number, e, the base of the natural logarithm.
- Euler's method, for finding approximate numerical solutions to differential equations.
- Euler's theorem on homogeneous functions, used in the differentiation of composite functions with partial derivatives.
- Euler's formula, $e^{ix} = \cos x + i \sin x$, for representing complex numbers in polar form, and Euler's identity, $e^{i\pi} + 1 = 0$.
- The Euler equation (or Euler–Cauchy equation), a linear differential equation of equidimensional form, $a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \ldots + a_2 x^2 y'' + a_1 xy' + a_0 y = 0$.
- Euler (or Euler–Bernoulli) beam theory for flexural deflections.
- The Euler load, $P = \pi^2 EI/L^2$, the compressive load for the first buckling mode (the deflection curve corresponding to the lowest critical load) in a slender homogeneous axially loaded column pinned at both ends.
- The Euler constant (or Euler–Mascheroni constant), γ, which appears in the representation of a Bessel function of the second kind of order zero as a linear combination of a Bessel function of the first kind of order zero and a series solution.

- Euler's theorem, which is related to rotations about different axes passing through a single point in rigid-body kinematics.
- Euler's equations of motion, $\mathbf{I}\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}\,\boldsymbol{\omega}) = \mathbf{M}$, central to three-dimensional rigid-body dynamics.
- The Eulerian angles, φ, θ, and ψ, which characterize the motion of a gyroscope.
- The Euler formulas, which give the Fourier coefficients for a Fourier series.
- The Euler transformation, for replacing an infinite series by another that converges faster.
- The Eulerian description of fluid motion, which represents fluid field properties in terms of space and time coordinates.
- The Eulerian derivative, the total derivative operator in fluid mechanics (alternatively known as the material derivative, particle derivative, Lagrangian derivative, or substantial derivative).
- Euler's turbine equation, for shaft torque in fluid turbomachinery analysis.
- Euler's equations of motion for a fluid field, the inviscid flow approximation to the Navier–Stokes equations.
- The Euler number, Eu, a dimensionless number which relates pressure to inertia forces as $\Delta p/\rho v^2$.
- The Euler equation (or Euler–Lagrange equation) used in the calculus of variations to determine the stationary values of functionals.
- Euler graph theory, applicable to hydraulic networks and computational meshes.

In developing his discoveries, Euler adopted a research approach that went something like this. He would begin by devising a solution to a specific problem, taking his ideas as far as they would go in the process. After writing about 10 papers on the topic, he would set it aside and focus on a different problem. 10 or 15 years later, he would revisit the issue, adding depth and generality by incorporating new knowledge, and leading to a clearer exposition of the original ideas. On reaching a dead end anew, he would drop the subject again and return to it some 10 or 15 years later. In this way, he contributed an enormous amount to the archival technical literature, with remarkably little repetition along the way.

So vast was Euler's scientific output that Truesdell (1) estimates about a third of all research publications on mathematics and mechanics between 1725 and 1800 being authored by him. His original papers continued being published fifty years after his death. By 1910, his publication count had reached 866. In addition to published work, Euler left behind over 3000 pages of mathematical notebooks, early drafts of books, and other manuscripts. The famous declaration by the prominent scientist and mathematician Pierre-Simon Laplace says it all:

'*Read Euler, read Euler, he is the master of us all!*'

Taking heed of Laplace's dictum, we find that one of Euler's manuscripts in particular holds a special place in the history of water-hammer.

XXXIII.

Principia pro motu sanguinis per arterias determinando.

(Exhib. 1775 Dec. 21.)

Figure 2. Title of Euler' paper presented in 1775

2 PRINCIPLES FOR DETERMINING THE MOTION OF THE BLOOD THROUGH THE ARTERIES

In 1911, the Swedish mathematician and historian Gustav Eneström produced a comprehensive index of Euler's known published works including books, journal articles, and some correspondences deemed to be significant. To each publication, an index number was assigned from E1 to E866, referred to by historians today as the 'Eneström number'. The complete index is available online (3).

Of these published works, the paper E855 (4) titled '*Principia pro motu sanguinis per arterias determinando*' (which translates as *Principles for determining the motion of the blood through the arteries*) is particularly noteworthy. It is an essay on fluid motion in elastic tubes, the first known attempted mathematical analysis of blood flow in arteries (5). Long before the paper was published in 1862 in Euler's collected posthumous works (*Opera Postuma* II, pp. 814-823), it was presented by Euler to the Petersburg Academy on the 21st of December 1775.

In addition to the extensive index, Eneström also compiled a collection of Euler's unpublished manuscripts. The list was incomplete, and about 1500 additional pages were found in the 1930s in the Incunabula Department of the USSR Academy of Sciences Library (1). The newly discovered material was painstakingly ordered by Gleb Mikhailov in the 1950s, who brought to light that the *Opera Postuma* had published in some cases only incomplete fragments of manuscripts. Indeed, E855 was one of these incomplete manuscripts, of which paragraphs 1 to 14 were missing in the 1862 publication.

Luckily, the missing paragraphs were found among the trove of new material. The complete paper was finally published in 1979 in the 16th volume of Euler's collected works (*Opera Omnia* II, 16, pp. 178-196). Five years earlier, Cerny and Walawender (6) had published an annotated translation of the incomplete paper. With access to the full version, Bistafa (7) has published a translation that is more faithful to the original Latin text.

In fact, it appears that Euler might have already worked on the subject prior to 1775. In 1742, he sent a submission to a competition that was launched by the Academy of Sciences in Dijon, France, as well as another submitted for the Paris Prize competition. The Dijon competition theme was '*déterminer la différence des vitesses d'un liquide qui passe par des tuyaux inflexibles et de celui qui passe par des tuyaux élastiques*' (or *to determine the difference in speeds of liquid flow in rigid tubes and in elastic tubes*').

Euler's Dijon paper seems to have been lost. On the 28th of August 1742, Euler wrote to Goldbach (6) (of Goldbach's Conjecture fame) as follows: '*Last March I sent a piece on the motion of fluids in elastic tubes to the academy of sciences in Dijon, which had set a prize of 30 Louis d'or on this subject*'. Euler had received no answer and feared that both his Dijon and his Paris entries were lost, writing '*which I very much regret for this reason only, that I have kept no copies of these two pieces*'. The Paris prize essay was not lost and was actually one of the three winners that year (paper E109).

The contents of Euler's Dijon paper remain a mystery. Truesdell (8) thought it unlikely to be similar to the Petersburg paper of 1775, as the latter employs concepts that Euler did not develop until 1755. As it turns out, this paper of 1775 contains the first appearance in print of the basic governing differential equations for water-hammer.

3 DISCUSSION OF EULER'S PAPER

3.1 Problem setup and model description (§§ 1–8)

Euler opens his paper with a discussion of the flow of blood through arteries. The heart is modelled as a single-cylinder piston pump, the artery as a straight tube '*since it is clear from Hydrodynamica that the curvature of tubes through which the fluid moves contributes scarcely anything to altering the fluid motion*', neglects the influence of small branches, and places an orifice at the tube discharge which, using empirical parameters, can be adjusted to model the flow resistance corresponding to the flow of blood in the vascular system. The pulse is represented by the reciprocal action of the pump.

He establishes Σ as the maximum cross-sectional area of the tube at infinite internal pressure. He then proposes a constitutive model for the cross-sectional area s,

$$s = \frac{\Sigma p}{c + p}$$

where p is the piezometric head and c is a constant representing tube wall compliance. For $p \to \infty$, $s = \Sigma$, and for $p \to 0$, $s = 0$ (the tube collapses).

Having proposed a functional relation between area and pressure, Euler states '*therefore our entire treatment brings us here, so that for any interval z and at any time t, we may be able to assign both the functions v [flow velocity] and p [piezometric head], to which end two equations must be investigated, one of which ought to be obtained from the continuity of the fluid flowing through the tube, and the other from the acceleration of the individual fluid elements arising from the pressing forces ... from which the whole motion of the fluid through such tubes should be determined*'.

3.2 Principle of continuity (§§ 9–11)

Euler proceeds to derive the area-averaged mass conservation equation for unsteady one-dimensional incompressible flow in an elastic tube and writes it out as

$$\left(\frac{ds}{dt}\right) + \frac{d(vs)}{dz} = 0$$

'*which is the equation that supplies us the principle of continuity*'. The first term is the change in local volume with time due to the elasticity of the tube, and the second term represents the instantaneous volumetric flux.

3.3 Principle of acceleration (§§ 12–17)

Next, Euler derives the linear momentum conservation equation for horizontal frictionless flow. '*According to the principle of Mechanica ... with g denoting the height that a weight falls in one second, the equation from the principle of acceleration will be*'

$$2g\left(\frac{dp}{dz}\right) = -v\left(\frac{dv}{dz}\right) - \left(\frac{dv}{dt}\right)$$

The left-hand-side term in this equation is the externally applied pressure gradient force. (The height fallen in one second is $gt^2/2 = g/2$, so that $2g$ in Euler's notation is equivalent to the acceleration of gravity g in modern notation). On the right-hand-side, the total time derivative of the velocity is composed of the convective acceleration and the local acceleration terms.

In paragraph 14, the last paragraph of the missing fragment, Euler concludes '*see then, both of our equations, which our principles of continuity and of acceleration have supplied, will hold as*

$$\left(\frac{ds}{dt}\right) + \frac{d(vs)}{dz} = 0 \tag{I}$$

$$2g\left(\frac{dp}{dz}\right) + v\left(\frac{dv}{dz}\right) + \left(\frac{dv}{dt}\right) = 0 \tag{II}$$

to which is added the formula

$$s = \frac{\Sigma p}{c+p}$$

and these should allow us to deduce what pertains to determining the motion of the fluid through the tube'.

Although Euler uses the symbol d, which today is used to denote ordinary differentials, it is clear from his manipulation of the variables that these are partial derivatives. Rewriting Equations (I) and (II) in modern notation with A denoting the cross-sectional area, H denoting the pressure head, and g the acceleration of gravity = 9.81 m/s^2,

$$\left(\frac{\partial A}{\partial t}\right) + \frac{\partial(vA)}{\partial z} = 0 \tag{1}$$

$$g\left(\frac{\partial H}{\partial z}\right) + v\left(\frac{\partial v}{\partial z}\right) + \left(\frac{\partial v}{\partial t}\right) = 0 \tag{2}$$

we see that Euler has successfully derived the differential equations governing the water-hammer phenomenon, known today as the "**classical water-hammer equations**", in this case describing the pulse.

In this analysis, Euler assumes constant fluid density but allows the cross-sectional flow area to vary. By recasting the equations in terms of s and v and eliminating the variable p, he combines the two equations (I) and (II) to get

$$s\left(\frac{dv}{dt}\right) - v^2\left(\frac{ds}{dz}\right) - v\left(\frac{ds}{dt}\right) + \frac{2gcs\Sigma^2}{(\Sigma-s)^2}\frac{d(s/\Sigma)}{dz} = 0$$

He then proposes an alternative logarithmic growth model for the cross-sectional area as

$$s = \Sigma\left(1 - e^{-p/c}\right)$$

which allows him ultimately to replace the last term in the equation by the simpler term

$$\frac{c\Sigma^2}{(\Sigma-s)^2}\frac{d(s/\Sigma)}{dz}$$

3.4 Rigid tube assumption (§§ 18–25)

Before moving on to the general fluid-structure interaction problem, Euler analyses the case where the tube is rigid. He does this in two different ways: first by directly integrating the differential equations, and then by using a work-energy approach.

The rigid-tube assumption means that the area *s* is always equal to the maximum area *Σ* and is a function of axial coordinate only. As such, $s \neq f(p)$ and $ds/dt = 0$, which allows Euler to eliminate the first term in the continuity equation, reducing the equation to its steady one-dimensional incompressible form $sv = bV$, where *b* is the cross-sectional area at another location and *V* is the velocity at that same other location, '*that is, at each location Z of the tube, the velocity v of the fluid will be inversely proportional to the amplitude [area] s of the tube, exactly as the theory of motion of fluids through tubes postulates*'.

By substituting terms, Euler develops an expression for pressure in the tube given *b*, *V*, *P* (piezometric head at specific point), and flow area as a function of distance along the tube:

$$2gp = 2gP + \frac{1}{2}V^2\left(1 - \frac{b^2}{s^2}\right) - \frac{bdV}{dt}\int\frac{dz}{s}$$

Euler now considers a constant-area pump attached to a tube, as in Fig. 3, with the pump initially filled and the tube initially empty. By a series of mathematical manipulations and introducing the downstream boundary conditions: *σ*, discharge orifice area, *π*, piezometric head at the discharge orifice, and *Ω*, total tube length, he arrives at the following equation

$$2g(p - \pi) = \frac{1}{2}b^2V^2\left(\frac{1}{\sigma^2} - \frac{1}{s^2}\right) - \frac{dV}{dt}(Z - \Omega)$$

'*from which the pressure [head] p of the fluid can easily be defined at any time at any location Z for any tube, and thus all things that pertain to the motion will be made known by this method*'. This is basically the one-dimensional incompressible unsteady Bernoulli equation involving the static pressure, kinetic energy, and acceleration terms.

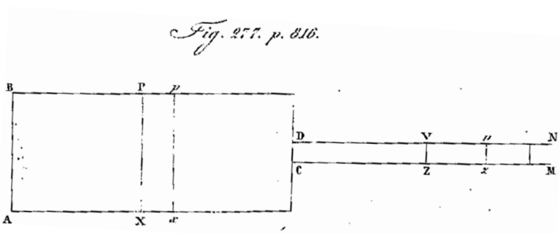

Figure 3. Euler' simplified pump-tube model of a heart attached to an artery

3.5 Uniform tube application (§§ 26–30)

Euler progresses further in the analysis by taking the case of a uniform tube with constant cross-sectional flow area. He derives closed-form expressions for the flow velocity and pressure in terms of the pump cavity size and discharge conditions. He then discusses difficulties he encounters in applying the direct integration approach to the case of elastic tubes and outlines the motivation for developing an alternative solution approach.

3.6 Alternative method for determining flow through rigid tubes (§§ 31–34)

In preparation for the general fluid-structure interaction problem, Euler re-derives the equations for flow in a rigid tube by using the work-energy principle (the '*method of living forces*'). '*And thus, we have now obtained through differentiation the same equation we had deduced above by integration. So, in this way it will also be convenient to consider the case of elastic tubes*'.

3.7 Elastic tube analysis (§§ 35–43)

He ends the paper with an attempted analysis of fluid motion through an elastic tube using the work-energy approach. He is unable to reduce the problem to one of integrating a single equation for a single unknown. This is not surprising to anyone familiar with this difficult fluid-structure interaction problem. The subject and its history are reviewed in (9).

Euler leaves us with the following parting words: '*since there is no clear way of accomplishing a solution, this research should be considered to transcend human powers, so that we are of course forced to put an end to our work here ... Thus, in solving for the motion of the blood, we encounter unconquerable difficulties that hinder the thorough and accurate exploration of all the works of the Creator; wherein there is much more supreme wisdom conjoined with omnipotence that we ought to admire and venerate, as not even the greatest human ingenuity avails to perceive and explain the true structure of the slightest micro-organism*'. On this point, he would turn out to be mistaken.

3.8 Summary of Euler's paper

The paper contains the first mathematical expression of conservation of mass for an incompressible liquid in an elastic tube, Leonardo da Vinci's pronouncement that '*by so much as you will increase the river in breadth, by so much you will diminish the speed of its course*' (10) notwithstanding.

By applying the principles of conservation of mass and linear momentum to unsteady one-dimensional incompressible flow through an elastic tube driven by a piston pump, Euler obtained the two governing differential equations (1) and (2) from the continuity and acceleration principles, respectively. (The second of these equations is known today as the one-dimensional form of the Euler equation for inviscid flow.)

When extended to include compressible fluids (i.e., variable density) these equations completely describe all the physics required to model water-hammer in a frictionless horizontal tube. In fact, in a treatise presented to the Petersburg Academy only a few months later, Euler himself describes an analysis in which the fluid density is explicitly formulated as a variable in the continuity equation (with the cross-sectional area taken as constant in time and variable along the axial length) (6).

4 HISTORICAL CONTEXT

The history of water-hammer in the 19th century has been presented in a number of papers in previous Pressure Surges conferences describing the discoveries of Johannes von Kries in 1883 (11, 12), Korteweg and Moens in 1878 (13), and Thomas Young in 1808 (14). Here, the preceding 18th century contributions of Euler are put in historical context and the development of the ideas is traced. But first, the issue of whether Euler was truly a 'pure mathematician' with no interest or ability in practical applications, as he is often portrayed in the popular literature, is addressed.

4.1 Practical hydraulics

4.1.1 The Sanssouci fountains

The origin of the pure mathematician Euler myth is undoubtedly a letter sent from King Frederick II to Voltaire on the 25th of January 1778 in which he wrote (2):

'*I wanted a water jet in my garden; the Cyclops Euler calculated the effort needed for the wheels to raise the water to a reservoir, from which it would fall back through canals,* *finally spouting at Sanssouci. My mill was built mathematically, and it couldn't raise a drop of water fifty feet from the reservoir. Vanity of Vanities! Vanity of mathematics*'.

The king wanted the fountain at his Sanssouci palace in Potsdam near Berlin, with water supplied from the River Havel. The critical design requirement was that the main fountain jet should be 100 feet high (30.5 m), so as to be more impressive than the fountains in Louis XVI's gardens in the palace of Versailles near Paris. O vanity of vanities, vanity of royal envy.

Interestingly, Peter the Great, who named the city of St. Petersburg after his patron saint, also entertained grand ideas of outdoing the Versailles fountains at Peterhof Palace. He passed away in 1725 however, before Euler was to move to Russia and have any opportunity to be involved in such state-sponsored projects.

4.1.1.1 Design and construction of the fountains

The Sanssouci fountains episode has been meticulously chronicled by Eckert (15, 16). The reservoir was about 1 km away from the river, at an elevation of about 50 m. Construction started in summer 1748, and by the end of the year the canal from the river to the pumping station, the windmill, and the pumps had all been completed. The pumps were connected to an alternative mechanism powered by horses too in case of no wind.

Figure 4. Fountains at the palace of Versailles (top photograph by first author) and Sanssouci (bottom photograph by second author)

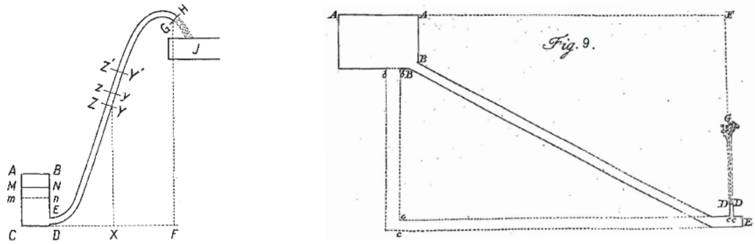

Figure 5. The reservoir project (left) and the fountain project (right)

The conduit between the pumping station and the elevated reservoir consisted of barrel-like wooden pipes of 7–9-inch diameter (~20 cm) reinforced with iron bands. Between the reservoir and the fountains, the diameter was 12–16 inches (~35 cm). The system was tested in spring 1749, but the pipes burst at the lower end near the pumps before the water was raised halfway up. The pipes were replaced with bored spruce tree trunks of 3–5-inch diameter (~10 cm) and tested again only to produce the same end result. Five 1.5-ton copper tanks (1524 kg) were placed along the pipeline, presumably to protect from pressure surge, but these were not effective.

4.1.1.2 Chronology of Euler's involvement

In the summer of 1749, the king sought Euler's help in designing the system hydraulics. The subsequent record of communications is as follows:

18 September 1749: Euler informed the president of the Berlin academy, Pierre Louis Maupertuis, that '*I sent my researches about the projected lottery yesterday to the King, and I hope to accomplish within a couple of days those about the hydraulic machine*'.

21 September 1749: Euler sent Maupertuis his first results on '*the Hydraulic Machine of Sans Souci*', in which he cautioned '*that it would require a huge effort to make [the fountain jet] as high as the King wishes*'. On September 27, Frederick II wrote to Euler confirming receipt of the preliminary investigations.

30 September 1749: Euler wrote to Maupertuis and identified some serious design flaws. Referring to Edme Mariotte's work on the fountains in Versailles, he stated that '*a lead pipe with a diameter of 12 inches [30.5 cm] and a wall thickness of [~3/16 in, 0.5 cm] is able to sustain a 100-foot-high [30.5 m] water column*', but cautioned against simple extrapolation, as the Sanssouci designer '*does not give any rule for estimating the pressure which the conduit pipes have to sustain. Apparently, he believes that these pipes would have to sustain the weight of the water column which corresponds to the state of rest*'.

Euler, an expert in hydrostatics, recognised that in addition to the hydrostatic pressure, the dynamic action of the pumps would result in unsteady fluid (rigid-column or water-hammer) forces that were not being considered in the design.

He recommended experiments on pipe wall thickness, '*for if Mariotte's experiment was imprecise, or corrupted by misprint, I would not know how to determine the thickness of the pipes for the situation on hand, other than doing new experiments on the force which the lead pipes can sustain. To trust mere chance in determining the pipe thickness would be taking too much risk*'.

It is notable that: (a) Euler assumed lead pipes would be used following the initial failure of 1749, yet the pipes continued to be made of wood, and (b) his explicit advice to do some testing was totally ignored. In referencing Bernard Forest de Bélidor and Edme Mariotte, Euler consulted the best available contemporary resources on hydraulics. It remained for him to analyse the pressure in the pipes as a result of the dynamic action of the pumps.

17 October 1749: Euler presented a summary of his results, along with some related windmill calculations, to the king. He wrote that, barring major revisions, the design as it stood would not work. The pumping pressure was too high, lead pipes were needed to replace the wooden pipes, and a reduced pressure would not deliver the water to the reservoir, '*for if one were unwilling to change anything in this regard, it would be almost impossible to raise more than 160 cubic feet per hour [4.5 m^3/hr]. Even for this, one would have to make considerable changes in the pump dimensions. In their current state, it is quite certain that one will never raise as much as one drop of water to the reservoir, and the entire force would be used only towards the destruction of the machine and the pipes*'.

Frederick II thanked Euler '*for the remarks you have made concerning your calculations about the pumps and pipes of the machine of Sanssouci. They have been very agreeable to me, and I am very obliged to you for the effort which you have contributed to it*'.

21 October 1749: Euler wrote to Maupertuis explaining his aversion to wooden pipes, '*the true cause of this irritating accident is solely that the pump capacity was too large, and if not reduced considerably, either by reducing the pump diameter, or height, or number of cycles per mill revolution, the machine will not be able to deliver a single drop of water to the reservoir*'.

23 October 1749: Euler presented a paper to the Berlin academy '*Sur le mouvement de l'eau par des tuyaux de conduite*' (or *On the motion of water in conduit pipes*, paper E206), in which he analysed unsteady fluid flow in a pipe and calculated the flow of water for various pump conditions and pipe dimensions. Although he does not explicitly mention Sanssouci in this paper, he uses the example of raising water to a reservoir by means of a pump to demonstrate the practical application of his theory.

Euler's analysis here foreshadows his famous formulation of the general equations of motion for ideal fluids (Euler's equations) published in 1755 in a paper titled '*Principes généraux du mouvement des fluides*' (or *General principles of fluid motion*). He successfully arrives at an equivalent to the unsteady Bernoulli equation from which the pressure can be determined. He also derives a mass-oscillation formula

$$p = g + \frac{0.256a^2bl}{c^2t^2}$$

where p is the piezometric head, g is the vertical height, a is the diameter of the pump, b is the height of the pump, l is the pipe length, c is the pipe diameter, and t is the duration of a single pump stroke. This formula contains the $t = 0$ singularity (infinite acceleration) known from rigid-column theory. The remarkable thing is that water-hammer theory is needed to "remove" this singularity. The time scale t is related to the system's excitation, here the dynamic action of the pump.

Not content with merely presenting the theory, he exhaustively calculates quantities of practical interest including the pressure near the lower end of the pipe and the quantity of discharge flow under various combinations of design parameters. He goes to great lengths to explain the practical significance of these quantities, specifically highlighting the added contribution of the dynamic pressure to the total pressure inside the pipe.

As far as the historical record shows, Euler was not to have any further involvement with the Sanssouci fountains project. His only other indirect contributions were a couple of papers presented on 20 November 1749 and 5 February 1750 titled '*Concerning different methods with which to raise water using pumps with the greatest efficiency*' and '*The most advantageous arrangement of machines used to raise water via pumps*', respectively.

4.1.1.3 Euler's recommendations

Euler presented his findings in his typical style which the mathematician and physicist Joseph Fourier described as '*that admirable clarity which is the principal character of all Euler's writings*'. He strove to be understood even by those not mathematically inclined, and disseminated his results in the form of simple rules:

- *For the same force acting on the pump pistons to deliver a maximum amount of water into the reservoir, one must make the rising pipe as wide as possible.*
- *To deliver a maximum quantity of water into the reservoir by the same force on the pistons, one must make the rising pipe as short as possible.*

He presented his calculations in tabular form, so that the practitioners (or 'fountaineers') could easily determine design parameters such as the appropriate pump size given the maximum pressure a pipe could withstand, '*because it is not always possible to make the rising tubes as strong as one wishes, in such cases one has to modify other parts of the machine, in order that nothing remains to be afraid of regarding the force in the tubes*'.

He did sample calculations to show the hazards of dynamic loading, where '*even though the height for raising the water was only 60 feet [18.3 m], the tube would have had to withstand a resultant force more than 5 times larger than the simple weight of the water column*'.

Baffling enough as it is that Euler's recommendations were all ignored, perhaps even more surprising is the fact that a hydraulic project of this scale was fully achievable with the 'rule-of-thumb' standards of practice that existed at the time, as exhibited most notably in the spectacular 'Marly Machine' at Versailles (17, 18), completed in 1688.

4.1.1.4 Completion of the Sanssouci fountains

In the summer of 1752, the designers finally decided to use lead pipes, but (against Euler's advice) the diameter was only 4 inches (10 cm), which produced a meagre flow to the reservoir. By spring of 1754, following a winter season of heavy precipitation, the king was given a demonstration with the reservoir half-filled. But the jet only rose to half the desired height, and the reservoir emptied within an hour. A new fountaineer was then handed the job, who, oblivious to the law of conservation of energy, proposed, among other things, to build the pipe in a U-shape so the flow would gain some energy on the way to the elevated reservoir.

The Seven Years' War broke out in 1756, and when work resumed in 1763, the plug was pulled on the project due to spiralling costs. In 1841, under the reign of a new king, the project was finally designed properly, with steam-engine driven pumps and 10-inch diameter (25 cm) cast-iron pipes, and the fountain project was successfully completed by 1843.

4.1.2 Fluid friction and ideal flows

It is also said of Euler, perhaps because his general equations treated idealized flows, that he did not appreciate the phenomenon of fluid friction and head loss in pipes. His two rules listed above, increasing the pipe diameter, and reducing the length, refute this claim. But also, in a paper he presented to the Berlin academy in 1751 and to the Petersburg academy in 1754, published in 1761, '*Tentamen theoriae de frictione fluidorum*' (or *An attempt at a theory of fluid friction*), he deals with the subject explicitly.

Euler guessed that the frictional forces in fluids arise from normal stresses (like in solids), rather than viscous stresses as we know today. Bistafa (19) has shown that this theory underestimates the head losses in a pipe. Nevertheless, it is clear that Euler was well aware of the need to account for frictional effects in hydraulic installations, and his conjecture was certainly plausible given the state of knowledge in his day.

Alas, for more than a hundred years after these pioneering attempts, the science of hydrodynamics and the art of hydraulics would develop along parallel paths, with sophisticated mathematical representations of idealized flows on the one hand, culminating in the Navier–Stokes equations, and empirical formulas for commonly encountered flow geometries on the other, yielding the Darcy–Weisbach equation. Fluid mechanics as a field was finally unified with Ludwig Prandtl's 20th century invention of boundary-layer theory.

4.1.3 Euler the fluids engineer

Far from being just a gifted mathematician (and among the most prolific of all-time), Euler was also deeply interested in practical problems and physical applications. In ballistics, he edited the German translation of Benjamin Robins' *New Principles of Gunnery* published in 1745, thereby placing the topic of artillery on a sound scientific base. His treatise on naval architecture *Scientia Navalis* published in 1749 introduced the concept of metacentric height and established the theory of ship stability as it is taught today. He created accurate geographic maps, dealt with building water canals, and improved the making of optical lenses, giving Germany a technological edge that would last for over a century.

In hydraulics, Euler developed the theory of pump and turbine design, correctly explained the concept of internal pressure, and gave criteria for avoiding cavitation. In 1754, he offered a detailed design for a guide wheel of a turbine. Jakob Ackeret built a model of it in 1944 (1) and found a respectable efficiency of 71 % compared to 78–82 % for the best turbines of similar capacity and head at the time. Euler's pragmatic (or 'engineering') sense is evident when he discusses a basic model for calculating the propelling force of the wind on the sails of a ship, '*until now the true theory has not been discovered. It should come as no surprise that in calculation we stay with this hypothesis, which we do not fail to recognise as being unsatisfactory*'. Euler reasons like any good engineer does today when faced with a theoretically intractable problem: it is better to have a rough approximation by making assumptions and understanding the limitations than to have no answer at all.

In summary, there is no evidence that the king's claims made in 1778 have any basis in reality. Euler, until the day he died, maintained a lively interest in real problems, many of which he solved ingeniously. On the day he died in fact, 7 September 1783, news had just arrived of the first manned ascent of the Montgolfier brothers' hot-air balloon, and the last calculation on Euler's slate dealt with how high the balloon would rise in the air.

Incidentally, in 1808, Joseph Montgolfier published an article (20) in which he proposed replacing the Marly Machine at Versailles with a water pump of his own design, the *Bélier Hydraulique* (or *Hydraulic Ram*), which worked on the principle of: water-hammer.

Figure 6. The Montgolfier hot-air balloon (courtesy of Wikimedia Commons)

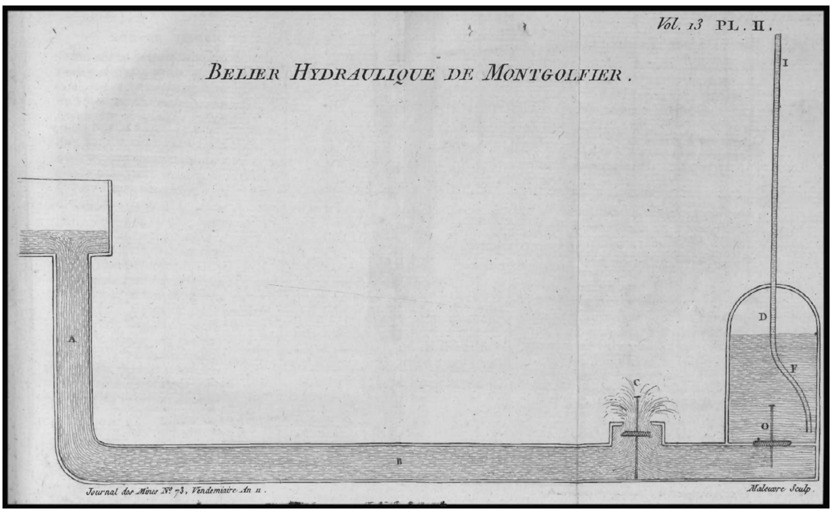

Figure 7. Hydraulic Ram (21): When the water in the pipe *AB* acquires a sufficient velocity, it raises the valve *C*, which stops its passage, so that a part of it is forced through the valve *O*, into the air vessel *D*, whence it rises through the pipe *F-I*

4.2 Theoretical hydrodynamics

The partial differential equations (PDEs) that Euler established in his paper of 1775 on the flow of blood in arteries involve rates of change of velocity and pressure in an elastic medium. Their solution requires knowledge of the calculus of differential equations for the rates of change and the theory of elasticity for the material behaviour.

What follows is a brief sketch of the state of knowledge in these areas at the time and an outline of some contemporaneous developments. The mathematics of discontinuous functions and general traveling-wave solutions to hyperbolic PDEs are particularly relevant, as well as the constitutive relationships in elasticity including the definition of a linear modulus of elasticity.

From our vantage point today, with over 250 years of accumulated knowledge in hindsight, it can be hard to appreciate just how extraordinary Euler's achievement was. Bear in mind that during his time, the ideas of Newtonian mechanics had not yet been widely accepted and were in fact bitterly contested, the distinction between mathematician and physicist did not yet exist, and the very notion that a computation could be mechanically automated had not yet been conceived (though Euler's method of numerical integration would become the basis for today's computer programs that do this).

Consider this case discussed by Rodriguez-Vellando in (22) for example. In his first paper on vibrations of elastic beams (paper E40, written in 1735), Euler integrates the ordinary differential equation for transverse vibration $y = k^4 y''''$. He does this using infinite power series. Only after four years of intense study does he realize that the solutions can be expressed using trigonometric and exponential functions. It is telling that it took this much effort for someone who is considered by many as the foremost mathematician of his generation to arrive at a mathematical discovery that seems so simple and rudimentary to us today.

4.2.1 Developments in the mathematics of wave mechanics

d'Alembert first obtained the partial differential equation for the vibration of a flexible string in tension in 1746 (23). It is known today as the one-dimensional wave equation. Euler took up the problem and in an essay on the propagation of sound ('*de la propagation du son*', paper E305) written in 1759, he derived the wave equation and provided the general solution which is known today as the traveling wave solution.

Euler's solution allowed for the representation of disturbances moving along a string at a constant speed, determined by the string's density and tension (or in the case of sound in air, moving at the speed of sound). d'Alembert opposed this approach, unconvinced that a mathematical function could take different expressions in different parts of its domain. In other words, he believed that a function must be everywhere smoothly continuous, while Euler proceeded to solve the wave equation using generalized discontinuous (or piecewise continuous) functions. In this way, Euler discovered the properties of wave propagation and reflection in the general solutions of hyperbolic partial differential equations, which are core elements of the water-hammer phenomenon (though Euler did not realise this).

This historical controversy over the nature of discontinuous change, at times acrimonious, has appeared in various other forms since, whether in biology (saltationism vs. evolution), in quantum mechanics (quantum jumps vs. quantum fields), in the philosophy of science (Kuhnian revolutions vs. Popperian incrementalism), and so on. In his paper on the propagation of sound, Euler hopes that '*in time, these conservative geometers will accept these new functions*'.

In this work, Euler arrives at a value of $\sqrt{gH}$ for the speed of sound in air. He commits the same error as Newton, who had also derived it, in relying on Boyle's law, which is valid only for equilibrium (isothermal) processes. The result deviates from experiments by about 20%. It was not until 1808 that Laplace, correctly identifying the process as adiabatic, introduced the specific heat ratio of 1.41 in dry air to arrive at a speed of sound value that matched experimental measurements.

Euler continued to work on the vibrating string throughout his career (see for example papers E213, E287, E317, E318) and he published his complete generalized solution to the wave equation in 1773, in paper E439. In this paper, he describes solutions to many specific problems involving shocks and corners. The history of later developments in this field is described in (24).

4.2.2 Developments in the theory of elasticity

Towards the end of Euler's life, the mathematical tools to solve hyperbolic PDEs had been developed. What remained was a theory of elasticity that could account for the elastic response of materials, such as liquid-filled elastic tubes. This would be achieved in the rigorous and systematic definition of the concept of the stress tensor by Cauchy in 1822 (25). And yet, Euler also had a major hand to play in the development of important aspects of the theory of elasticity.

In paper E831, Euler derives the formula for the bending moment acting on a slender homogeneous beam, $M = EI/r$. His approach is basically identical to that used to introduce engineering students to the subject in modern textbooks. To do this, Euler needed to define the linear modulus of elasticity for a Hookean material, today referred to as 'Young's Modulus'. In 1776, he wrote a paper (E508) on the bearing loads of columns, in which he definitively discussed the concept of a linear modulus of elasticity and its relation to the bending moment acting on a beam.

4.2.3 Developments in the theory of water-hammer

Although Euler presented his water-hammer equations in 1775, he does not seem to have directly influenced later researchers in the field, likely because his ideas were not published in complete form for over two hundred years, until 1979. The classical water-hammer equations were derived again in the 1850s, presumably independently, by the scientist Ernst Weber, and they were published by his brother Wilhelm Weber in 1866 (26). At the time, the topics of wave propagation in compressible fluids in rigid tubes and incompressible fluids in elastic tubes were treated as two separate and distinct disciplines. The first to consider wave propagation of a compressible fluid in an elastic tube was Diederik Korteweg in 1878 (27). That history is relayed in (13) and (28).

5 CONCLUDING REMARKS

In his remarkable paper of 1775, partly published in 1862 and fully published in 1979, Euler develops a one-dimensional model of blood flow through an artery driven by a heart in the form of a piston pump. He arrives at the system of hyperbolic partial differential equations that governs the behaviour of water-hammer. He includes the effects of flow area variation and wall deformation in his analysis. To the extent that he is unsuccessful in fully solving the problem, he is held back by the state of development in the theories of mathematics and elasticity of his time.

Despite being highly familiar with solutions for wave propagation by nature of his extensive research and discoveries on the subject, Euler shows no signs in this paper of recognising the wave nature of the equations. Furthermore, again despite being the first to describe a linear modulus of elasticity in his work on the elastic curvature of beams, he does not establish an adequate constitutive relationship in this paper to model the behaviour of the tube wall, understandable given the viscoelastic material nature of artery walls.

Euler's trailblazing work on the propagation of sound waves in air and elastic waves in strings allowed him to establish the general form of the solution to the one-dimensional wave equation. His pioneering work in elasticity enabled him to establish the significance of a linear modulus of elasticity for describing elastic deformations of structures. With these tools at his disposal, his inability to progress beyond a rigid tube in the problem he set out for himself on the flow of blood in arteries might be considered as somewhat of a missed opportunity. Nevertheless, despite the primitive state of development in the mathematics and mechanics of his era, he single-handedly derived the equations that govern the behaviour of water-hammer. This in itself is a notable achievement that is worthy of our utmost respect and admiration.

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