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On Boussinesq and Coriolis coefficients and their derivatives in transient pipe flows

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ABSTRACT

A rigorous analytical justification is developed for a simplification that is widely used in one-dimensional simulations of steady and unsteady flows in pipes, namely treating the flow as a 'plug flow' in which cross-sectional variations in axial velocity are neglected except for consequential shear forces on pipe walls. The proof is obtained without assuming that the flow is nearly incompressible flow and, indeed, it is found that the plug flow approximation remains good even for compressible flows at moderate, subsonic speeds. In more general analyses, explicit allowance is sometimes made for the influence of (i) the Boussinesq coefficient β and (ii) its axial rate of change $\partial\beta/\partial x$. Typically, such analyses discard the terms in $\partial\beta/\partial x$ in the basic equations and proceed using β alone. This is done on the grounds that no method of evaluating $\partial\beta/\partial x$ is available. It is shown in this paper that discarding $\partial\beta/\partial x$ is not only unnecessary, but that it actually leads to less accurate outcomes than simply assuming plug flow. The process used to derive the analytical justification has a spin-off benefit of shedding light on alternative methods of integrating source terms over the pipe cross-section. Although the primary purpose of the paper is to demonstrate that the use of plug flow approximations can be justified rigorously for most flows, brief attention is also paid to cases where it has potential to mislead.

1. Introduction

In most analyses of fluid flows along pipes, it is usual to assume *uniaxial* conditions. That is, all (local-mean) velocity vectors are assumed to be parallel to the axis of the pipe, herein chosen as the x-axis. Nevertheless, although all velocity vectors are assumed parallel, they are not necessarily assumed equal. Indeed, except for special cases such as superconducting flows at extremely low temperature, the velocity in real pipes ranges from zero at the pipe wall to a maximum somewhere within the cross-section. In effect, the overall flow resembles numerous small stream-tubes, all parallel to the axis, but each with its own velocity of flow. The mean velocity U of the bulk flow is somewhere between zero and the maximum velocity. Although the velocity varies laterally as well as axially, the flow is termed 'uniaxial', not 'two-dimensional' or 'three-dimensional', because these terms are commonly reserved for flows in which the directions of the velocity vectors

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Nomenclature

A	cross-sectional area – m^2
c	sound speed – m/s
F_τ	shear force per unit length, N/m
g	gravitational acceleration – m/s^2
h	pressure head – m
LHS	left-hand side of equations
M	Mach number
MOC	method of characteristics
p	pressure – Pa
RHS	right-hand side of equations
t	time coordinate – s
U	mean velocity – m/s
u	local velocity – m/s
x	distance coordinate – m
Greek symbols	
α	Coriolis coefficient – see Eq. (1.3)
β	Boussinesq coefficient – see Eq. (1.2)
Δ	finite interval
λ	coefficient defined in Eqs. (8.3), (8.6) & (8.9)
ρ	fluid density – kg/m^3
τ	shear stress – Pa
Suffices	
w	wall

are not parallel to a single axis. If the density does not vary over the cross-section, the mean velocity satisfies

$$AU \equiv \int_A u dA \tag{1.1}$$

in which A denotes the overall cross-sectional area and u is the velocity in a typical stream-tube of area δ_A. The product AU is a volumetric flowrate Q and this is often used as the main flow-dependant variable instead of the mean velocity. This substitution can be advantageous, especially pipelines for which $A = A(x)$ is not a constant and when implementing continuity conditions at boundaries. Nevertheless, for the purposes of this paper, the use of a mean velocity is more convenient because the focus is on consequences of variations in axial velocity in the flow cross-section. These can become influential when account needs to be taken of powers of the velocity, e.g. u^2 and u^3 in the momentum and energy equations. In this case, integration over the cross- section leads to

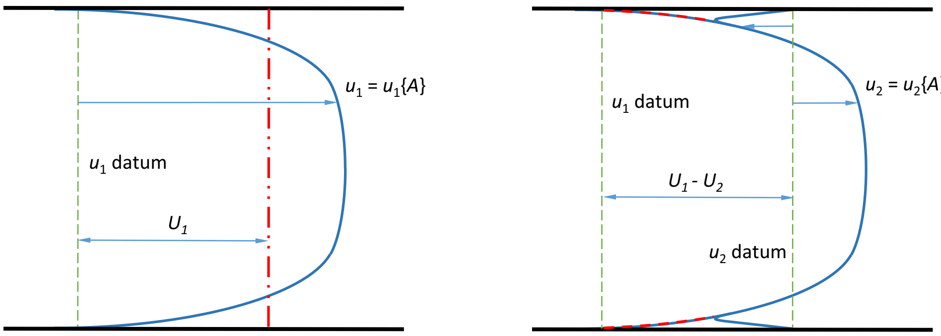
$$\beta AU^2 = \int_A u^2 dA \tag{1.2}$$

and

$$\alpha AU^3 = \int_A u^3 dA \tag{1.3}$$

where β and α are known as the Boussinesq and Coriolis coefficients and, in practice, these are always greater than one (*NB: It is always true that the square of the mean of a set of non-equal numbers is smaller than the mean of their squares*). For example, in a *steady laminar* flow of a Newtonian fluid in a pipe of circular cross-section, $\beta = 4/3$ and $\alpha = 2$. However, this is a rare, special case for which the variation of velocity over a cross-section can be determined analytically. In most cases, including all turbulent flows, cross-sectional variations cannot be known a priori and cannot be determined during the course of a wholly one-dimensional (x,t) analysis – even though the general flow behaviour might be known. For instance, eighty years ago, Streeter [1] presented graphs showing the dependence of β and α on Reynolds number in steady flows along pipes and wide channels. It is found, for example, that in *steady turbulent* flows of Newtonian fluids in pipes of uniform cross-section, the numerical values can be in the order of $\beta \approx 1.02$ and $\alpha \approx 1.06$. In engineering practice, it is commonly judged that such values are sufficiently close to unity for them to be approximated as unity. For completeness it is noted that there are similar correction factors related to the turbulent *fluctuations* of velocity and density [2,3].

Neglecting differences from unity in unsteady flows is less easily justified, especially in rapidly-changing, wave-like flows. In such cases, any value greater than unity is possible. Large values routinely occur when sudden wavefronts cause flow



(a) Initial velocity distribution (b) Distribution after step change

Fig. 1. Flow reversal within a cross-section caused by a sudden wavefront.

reversals in part, but not all of a flow cross-section (e.g. the step change in mean velocity from U_1 to U_2 depicted in Fig. 1). In the particular case of zero net flow, the values become infinite – see also [4]. Flow reversals such as this can be caused, for instance by the rapid closure of a valve.

Common sense tells us that these extreme conditions do not occur sufficiently often for their worst consequences in uniaxial models with variable β and α to be of widespread significance. If this were not so, significant differences between typical predictions and measurements would be more commonplace. Nevertheless, it is somewhat surprising that so little attention is paid to the influence of β and α in flows in which extreme values and singularities may be expected – as, for instance, in studies of skin friction close to sudden wavefronts. It is shown below that serendipity probably has a role to play in this respect.

1.1. Outline of paper

One purpose of this paper is to assess how strongly β and α influence different methods of analysing pressure transients. First, Section 2 summarises assumptions underlying Sections 3–5. The latter present formulations of the continuity and momentum equations based on three different standpoints, all of which are to be found in reputable papers published in journals and conferences. In some of the formulations, derivatives of β and α inevitably arise, and these are even more difficult to quantify accurately than the absolute values of β and α themselves. Typically, analysts simply neglect the derivatives, but give little or no formal justification for doing so. Then, in Section 6, it is shown that one of the derivatives, namely $\partial\beta/\partial x$ can be eliminated rigorously from the governing equations and that, when this is done, a simplification that is widely used in one-dimensional methodologies falls out naturally. The new analysis, designated as enhanced-area-weighted approach, may be regarded as analytically exact, subject to declared limitations on the types of flow for which it is intended.

The remainder of the paper begins with a discussion of source terms that feeds into the development of the Method of Characteristics based on the three formulations of the continuity and momentum equations (Section 8). Consideration is given in Section 9 to some specific flow cases and the relevance of the work to two particular categories of unsteady flow is then discussed (Section 10) and the main conclusions to be drawn from the paper are summarised (Section 11). A simple, but important, theoretical result is presented in Appendix-1 and the significance of the analysis for a range of flow conditions is quantified in Appendix-2.

It is important to state that the paper is targeted at transient flows of the type familiar to analysts of pressure waves associated with phenomena such as water hammer. When appropriate (namely, for fluids of sufficiently small compressibility), the density may be interpreted as a constant. Herein, however, derivatives of the density are not treated as zero. Indeed, the derivation of the MOC equations assumes an explicit relationship between the derivatives of density and pressure, as does the determination of wave speeds. It is also useful to note that, although assumptions made in the development approximate closely to reality during periods of rapid change (e.g., the passage of short wavefronts), they are less suitable for extended periods that usually exist between such wavefronts. The reasons for this are discussed in Section 2 considering different categories of unsteady flow.

It is conventional for a summary of related work by previous authors to be reported early in a paper. In this instance, however, the summary is deferred until after the various formulations of the governing equations have been presented. This enables direct cross-correlations to be discussed between the open literature and the current development.

2. Flow states

Mathematically, all core equations herein are expressed in a spatially one-dimensional form using (x,t) coordinates, in which the x-axis coincides with the pipe axis and t denotes time. Physically, however, cross-sectional variations exist in all parameters, the most obvious being the velocity, which ranges from zero at a stationary pipe wall to a maximum somewhere in the cross-section.

That is, the *physical* velocity field cannot be described fully without making allowance for at least one more spatial coordinate. This contradiction is overcome by using cross-sectional averages of parameters in the equations – e.g. the mean velocity U defined in Eq. (1.1). The use of this simplification has huge practical advantages, but, for purists, it is also the fundamental cause of the need to introduce the parameters β and α defined in Eqs. (1.2) & (1.3) Furthermore, other simplifications of lesser significance are almost always made without even declaring them. For example, in each of Eqs. (1.1)–(1.3), it is implicitly assumed that the fluid density does not vary over the cross-section, even though tiny variations will necessarily exist because of the influence of gravity. The remainder of this Section is devoted to clarifying some, but far from all, implicit assumptions made herein.

2.1. Velocity distribution

In the simplest case considered, *plug flow* conditions are assumed. That is, the velocity everywhere in any cross-section is deemed to be equal to the mean velocity. Thus, for instance, in Fig. 1(a), all $u_1 = U_1$ and in Fig. 1(b), all $u_2 = U_2$. This is the most common case assumed in practical simulations. In all other cases, the existence of cross-sectional variations in velocity is assumed and this leads to $\beta > 1$ and $\alpha > 1$. In the remainder of the paper, the term *uniaxial flow* is used to denote this more general case.

2.2. Fluid properties

The analysis presented below is valid for gas flows as well as liquid flows. In both cases, however, no account is taken explicitly of cross-sectional variations of any parameter except the velocity even though, for instance, variations in pressure and density will exist because of gravity. The values of pressure p and density ρ used in the equations may be interpreted with sufficient accuracy as cross-sectional averages. However, it is assumed that *changes* in these variables satisfy $c^2 = \partial p/\partial \rho$, where c denotes the isentropic speed of sound.

2.3. Steady and unsteady flow

The primary focus of this paper is on unsteady flows that include wave-like behaviour. Nevertheless, the equations presented are also valid for flows that vary too slowly for wave action to be discernible, and they are even valid for steady flows. In all cases, the possibility of axial changes in density, however small, is retained. Thus, for instance, if finite wall shear-forces exist in a steady flow along a horizontal pipe, the density will decrease slowly downstream and the velocity will increase – because continuity requires that the product 'density $\times$ velocity' is constant in a steady flow (e.g. [5]). Broadly similar behaviour will exist in accelerating flows where the velocity is almost uniform along a pipe. That is, the conditions will not be identical to those that are predicted by methods based on so-called rigid-column theory in which changes in density are excluded. Often the differences will be too small for the distinction to be of importance for engineering design purposes. Nevertheless, allowing for finite values of derivatives of the density enables the analysis to be used for steady or unsteady flows regardless of whether compressibility plays an important role.

2.4. Inertial and viscous causes of changes in β and α

In *steady* flows of real fluids, the velocity distributions in cross-sections sufficiently far from upstream and downstream boundaries depend strongly on viscous forces. When rapid transients such as those experienced in water-hammer occur, the velocity distributions change in manners such as those depicted in Fig. 1 in time scales that are tiny in comparison with viscosity-induced behaviour. This paper focusses exclusively on these rapid changes induced by inertial forces during such transients. No attention is paid to periods between successive transients, during which viscous forces dominate the relaxation of the newly-formed profiles back towards quasi-steady distributions typical of steady flows.

3. Plug flow formulation

The three formulations of the continuity and momentum equations are now derived from first principles, beginning with the ideal case of plug flow in which the velocities at all points in the cross-section are equal. The continuity and momentum equations for a pipe of uniform cross-sectional area A may be written as

Continuity (plug flow, full area):

$$A\frac{\partial \rho}{\partial t} + AU\frac{\partial \rho}{\partial x} + \rho A\frac{\partial U}{\partial x} = 0 \tag{3.1}$$

Momentum (plug flow, full area):

$$A\frac{\partial p}{\partial x} + \rho A\frac{\partial U}{\partial t} + \rho AU\frac{\partial U}{\partial x} = F_{\tau,w} \tag{3.2}$$

in which p denotes the fluid pressure and F_τ, the shear force per unit length acting on the fluid, is defined here as positive in the positive x-direction. The additional suffix w is used as a reminder that, for the conditions assumed in the plug flow formulation, F_τ exists only at the surface of the pipe wall. For clarity, the only source term considered is frictional resistance. In the absence of other source terms, A is redundant in Eq. (3.1), but it is retained in the interests of clarity in subsequent sections.

For simplicity, no account is taken of complicating factors such as non-uniform pipes/channels or continuous mass addition/removal. However, these equations and all subsequent equations are presented in forms that are applicable to any subsonic flow of a compressible fluid as well as for low Mach-number flows of liquids of low compressibility. To facilitate this, the fluid pressure is *not* substituted by the so-called pressure head $h = p/\rho g$, where g denotes gravitational acceleration. It is noted in passing that, strictly, the existence of the shear stress term in Eq. (3.2) is incompatible with the assumption of a uniform velocity distribution in the cross-section. However, this formulation of the equations is used in water-hammer analyses, so it is useful to include it in the comparisons presented herein. Also, the objection does not apply below when these equations are used to describe flows in elemental stream-tubes. In those cases, however, the mean velocity U needs to be replaced by the local velocity u applicable to any particular stream-tube.

In the case of simulations of transient liquid flows in pipes, it is common to discard the convective terms in these equations – i.e., $U\,\partial\rho/\partial x$ and $\rho U\,\partial U/\partial x$ – on the grounds that these are much smaller than $\partial\rho/\partial t$ and $\rho\,\partial U/\partial t$ respectively (e.g. [6,7]).

However, in the following development, these terms are not discarded. In part, this is because the purposes of this paper are not restricted to liquid flows. In part, it is because discarding the terms would prevent meaningful discussion of the roles of β and α and would hide implications of further assumptions that are commonly made when considering uniaxial forms of the equations.

4. Area-weighted formulation

To develop the more general *uniaxial* expressions, the equations are first expressed for individual stream-tubes of elemental cross-sectional area δ_A and are then integrated over the entire cross-section A. Each stream-tube has its own velocity u so the integrations over the cross-section involve terms such as those on the right-hand sides of Eqs. (1.1) & (1.2). In general, the pressure and density also vary over the cross-section, but this additional complication is neglected herein. For completeness, however, note that non-uniform pressure distributions are common at boundaries (e.g., at junctions, bends and valves) and lateral density gradients are a fundamental feature of some stratified flows. The equations for an elemental stream-tube can be expressed as:

Continuity (stream-tube):

$$\frac{\partial\rho}{\partial t}\delta A + u\frac{\partial\rho}{\partial x}\delta A + \rho\frac{\partial u}{\partial x}\delta A = 0 \tag{4.1}$$

Momentum (stream-tube):

$$\frac{\partial p}{\partial x}\delta A + \rho\frac{\partial u}{\partial t}\delta A + \rho u\frac{\partial u}{\partial x}\delta A = \delta F_\tau \tag{4.2}$$

where δF_τ is the net shear force per unit length on the lateral surfaces of the stream-tube. The distribution of this force around the stream-tube circumference will typically be strongly non-uniform. At any particular location on the circumference, the *direction* of the force will depend on whether the adjacent stream-tube is moving faster or more slowly.

To integrate these relationships over the whole cross-section, it is necessary to choose how the contributions of the various stream-tubes should be weighted. Two methods are used in the literature. In the one presented in this Section, the contributions are weighted in proportion to their cross-sectional areas and so integration over the whole cross-section (regarding p and ρ as uniform over the cross-section), leads to:

Continuity (uniaxial, area-weighted):

$$\frac{\partial\rho}{\partial t}\int_A dA + \frac{\partial\rho}{\partial x}\int_A u dA + \rho\int_A \frac{\partial u}{\partial x}dA = 0 \tag{4.3}$$

Momentum (uniaxial, area-weighted):

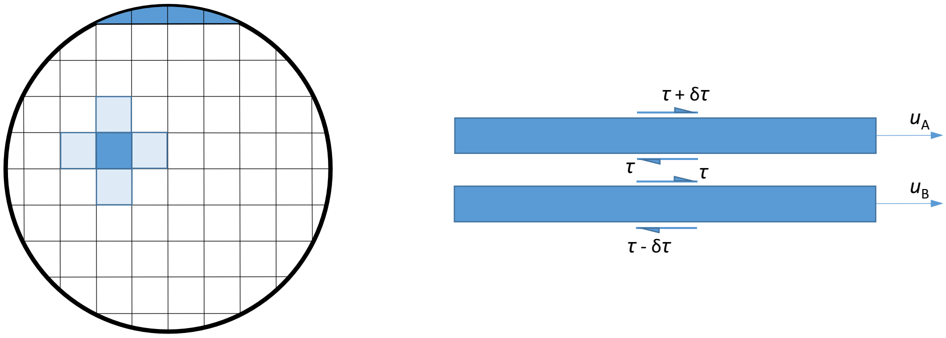
$$\frac{\partial p}{\partial x}\int_A dA + \rho\int_A \frac{\partial u}{\partial t}dA + \rho\int_A u\frac{\partial u}{\partial x}dA = \int_A \frac{\partial F_\tau}{\partial A}dA \tag{4.4}$$

Since the integration over the cross-section is independent of x and t, use may be made of integrations such as:

$$\int_A \frac{\partial u}{\partial x}dA = \frac{\partial}{\partial x}\int_A u dA = A\frac{\partial U}{\partial x} \tag{4.5}$$

and

$$\int_A u\frac{\partial u}{\partial x}dA = \int_A \frac{1}{2}\frac{\partial u^2}{\partial x}dA = \frac{1}{2}\frac{\partial}{\partial x}\int_A u^2 dA = \frac{1}{2}A\frac{\partial}{\partial x}\left(\beta U^2\right) = \beta AU\frac{\partial U}{\partial x} + \frac{1}{2}AU^2\frac{\partial\beta}{\partial x} \tag{4.6}$$



(a) Discretisation of cross-section as multiple, square-shaped stream-tubes

(b) Shear stresses on adjacent stream-tubes

Fig. 2. Indicative geometrical discretisation of flow along a pipe.

Accordingly, after dividing each term by the area A, Eqs. (4.3) and (4.4) can be expressed as:
Continuity (uniaxial, area-weighted):

$$\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x} + \rho\frac{\partial U}{\partial x} = 0 \tag{4.7}$$

Momentum (uniaxial, area-weighted):

$$\frac{\partial p}{\partial x} + \rho\frac{\partial U}{\partial t} + \beta\rho U\frac{\partial U}{\partial x} + \frac{1}{2}\rho U^2\frac{\partial \beta}{\partial x} = \frac{1}{A}\int\limits_A \frac{\partial F_\tau}{\partial A}dA \tag{4.8}$$

The equations simplify considerably if the convective terms are omitted from the original Eqs. (3.1) and (3.2). In that case, the second term in Eq. (4.7) vanishes, as do the third and fourth terms in Eq. (4.8) – i.e., both of the terms that include the Boussinesq coefficient.

4.1. Comparison of plug flow and area-weighted formulations

A comparison of the left-hand sides of Eqs. (4.7) & (4.8) with those of Eqs. (3.1) & (3.2) shows that the continuity equations are identical, but that the momentum equations differ in the following respects:

$\frac{\partial p}{\partial x}$	No change - because p is assumed to be uniform even though u is non-uniform
$\rho\frac{\partial U}{\partial t}$	No change - because ρ is assumed to be uniform even though u is non-uniform
$\beta\rho U\frac{\partial U}{\partial x}$	This term can be regarded as the corresponding plug flow term weighted by the Boussinesq coefficient. It is commonly assumed that this is the only change that needs to be made to the plug flow formulation.
$\frac{1}{2}\rho U^2\frac{\partial \beta}{\partial x}$	This term does not exist in the plug flow formulation. It is commonly discarded in numerical simulations that are nominally regarded as uniaxial. This is equivalent to assuming that $\delta U^2/U^2 >> \delta\beta/\beta$, but the authors do not recall seeing any mathematical justification for this assumption. Instead, it appears to be made simply because there has been no known way of evaluating $\partial\beta/\partial x$ without explicit information about the velocity distribution that cannot be available in a 1-D analysis. This paper provides a rigorous method of treating this term and shows that it leads to a simplification that provides new insight into the true role of the Boussinesq coefficient in both steady and unsteady flows.

At first sight, another difference between the plug flow and area-weighted uniaxial formulations exists, namely the source terms on the right-hand sides of the momentum equations. In practice, however, the two expressions are exactly equivalent even though one requires integration over the cross-section and the other does not. Indeed, the source term integral in Eq. (4.8) can be undertaken in a general manner without any knowledge whatsoever of the actual distribution of velocity or shear stress over the cross-section. For this purpose, it is useful to envisage the cross-section as a collection of stream-tubes, not necessarily infinitesimal (Fig. 2a). Further, for descriptive purposes, it is convenient to choose rectangular sections so that all surfaces of the tubes except those in contact with the pipe surface are flat. First, consider a typical internal stream-tube such as the one highlighted in Fig. 2a. The net shear force acting on this tube is the sum of the forces on its four lateral faces. Likewise, the net shear force on any one of the four adjacent tubes is the sum of the lateral forces on its faces. However, when the combined sum of any pair of adjacent stream-tubes is evaluated, the contributions of the common face cancel each other because, as illustrated in Fig. 2b, the shear stresses on the two tubes are in opposite directions. When the summation is made for all tubes, the contributions on all of the internal surfaces cancel and all that remains is the sum of the shear forces on the pipe surface, as illustrated by the highlighted outer tubes in Fig. 2a. Accordingly, in the following development, the area-weighted momentum equation is treated as

Momentum (uniaxial, area-weighted):

$$\frac{\partial p}{\partial x} + \rho\frac{\partial U}{\partial t} + \beta\rho U\frac{\partial U}{\partial x} + \frac{1}{2}\rho U^2\frac{\partial \beta}{\partial x} = \frac{F_{\tau,w}}{A} \tag{4.9}$$

This is an example of Stokes' theorem [8] so, in principle, the above paragraph could simply be assumed. However, it is included partly for completeness and partly as a fore-runner to the corresponding case in the flow-weighted formulation for which Stokes' theorem is not applicable.

5. Flow-weighted formulation

Some researchers (e.g. [9–12]) favour an alternative development of the uniaxial equations in which cross-sectional averaging is undertaken by weighting the influence of each stream-tube in proportion to its contribution to the total flowrate instead of in proportion to its contribution to the total area. Mathematically, both approaches are valid, but they are not equivalent. Instead, they provide information about different flow properties. For example, the area weighted formulation of the momentum equation provides information about force and momentum whereas the flow-weighted formulation provides information about power and energy.

Using the flow-weighted approach, the uniaxial equations for a stream tube are obtained by multiplying every term in Eqs. (4.1) & (4.2) by the local velocity, giving:

Continuity (stream-tube, flow-weighted):

$$u\frac{\partial \rho}{\partial t}\delta A + u^2\frac{\partial \rho}{\partial x}\delta A + \rho u\frac{\partial u}{\partial x}\delta A = 0 \tag{5.1}$$

Momentum (stream-tube, flow-weighted):

$$u\frac{\partial p}{\partial x}\delta A + \rho u\frac{\partial u}{\partial t}\delta A + \rho u^2\frac{\partial u}{\partial x}\delta A = u\delta F_\tau \tag{5.2}$$

On integrating these in a similar manner to above, we obtain (after dividing each term by the overall flowrate AU):

Continuity (uniaxial, flow-weighted):

$$\frac{\partial \rho}{\partial t} + \beta U\frac{\partial \rho}{\partial x} + \beta\rho\frac{\partial U}{\partial x} + \frac{1}{2}\rho U\frac{\partial \beta}{\partial x} = 0 \tag{5.3}$$

Momentum (uniaxial, flow-weighted):

$$\frac{\partial p}{\partial x} + \beta\rho\frac{\partial U}{\partial t} + \frac{1}{2}\rho U\frac{\partial \beta}{\partial t} + \alpha\rho U\frac{\partial U}{\partial x} + \frac{1}{3}\rho U^2\frac{\partial \alpha}{\partial x} = \frac{1}{AU}\int_A u\frac{\partial F_\tau}{\partial A}dA \tag{5.4}$$

Once again, the equations simplify considerably if the convective terms are omitted from the original Eqs. (3.1 and (3.2). In that case, the second term in Eq. (5.3) vanishes, as do the fourth and fifth terms in Eq. (5.4) – i.e., both of the terms that include the Coriolis coefficient. Nevertheless, in contrast with the area-weighted formulation, the derivatives of β will remain even if the convective terms are omitted. In practice, they will be strongest in regions of rapidly varied flow and weakest in regions of quasi-steady flow. Further attention is paid below to the dependence on the overall nature of the flow (see Section 9).

5.1. Comparison of area-weighted and flow-weighted formulations

The flow-weighted Eqs. (5.3) & (5.4) are significantly more complex than the area-weighted Eqs. (4.7) & (4.9) even though both sets must have exactly the same solutions. The continuity equation includes both β and $\partial\beta/\partial x$ and the corresponding momentum equation includes β and $\partial\beta/\partial t$ (although not $\partial\beta/\partial x$) and it also includes α and $\partial\alpha/\partial x$. Furthermore, the source term on the right-hand side of Eq. (5.4) cannot be evaluated as easily as its counterpart in the area-weighted formulation. Indeed, it cannot be evaluated at all without direct information about the (inherently unknown) distributions of velocity and shear stress over the cross-section.

At first glance, it might be expected that, once again, only the values at the pipe surface would contribute to the integral. However, it is easy to see that this cannot be the case because the velocity at the boundary is zero and so the integral would also be zero. The reason for the apparent paradox is that the weighting factors u_A and u_B in any typical pair of adjacent stream-tubes A and B are different. Therefore, although the magnitudes of the shear stresses at a common interface are the same for both tubes, the product 'velocity × shear-force' is not the same. This is true even for elemental stream-tubes whenever a velocity gradient exists. It follows that the value of the overall integral depends on the state of flow over the whole cross-section, not only on its state adjacent to the wall surface. *[As an aside, it seems worth noting that this difference has caused long-standing confusion in the pressure-surges community. Some researchers have argued that the shear terms cannot be evaluated without a knowledge of the internal flow state whereas others have asserted that it is sufficient to know the shear force at the wall. In fact, as pointed out by Brunone* et al. *[10], both groups are right – the first in relation to the flow-weighted formulation and the second in relation to the area-weighted formulation].*

Another disadvantage of the expression on the right-hand side of Eq. (5.4) is that it becomes a singularity when the mean velocity is zero – or indeterminate if the integral is also zero. This complicates uses of the equation. The complications are especially severe in numerical simulations because they will exist when the mean velocity is very small during flow reversals of the type illustrated in Figure 1.

6. Enhanced area-weighted formulation

A major practical limitation of both of the above formulations of the uniaxial equations is that they cannot be solved without making essentially arbitrary assumptions about β and $\partial\beta/\partial x$. Most analysts simply avoid the issue by using the plug flow formulation, namely, to assume $\beta = 1$ and $\partial\beta/\partial x = 0$. The most common approach of analysts who wish to make some allowance for the velocity distribution is to choose an estimated value for β, but nevertheless to assume $\partial\beta/\partial x = 0$. This is a reasonable approach in regions of steady or quasi-steady flow, but it is difficult to justify in regions of rapidly-varied flow close to pressure wavefronts. It is now shown that this problem can be reduced greatly by making use of information from a comparison of the two formulations. Specifically, the uniaxial momentum equation can be expressed in a form that does not contain any derivative of β. Furthermore, the resulting equation is shown to approximate closely to the plug flow equation.

Although the area-weighted and flow-weighted approaches yield different equations, they are mathematically compatible provided that the same assumptions are used in both cases – e.g., regarding the inclusion or omission of the convective terms. Therefore, meaningful results can be obtained by expressing differences between them mathematically. In particular, consider the two continuity equations. Subtraction of Eq. (4.7) from Eq. (5.3) yields

Continuity (flow & area formulations):

$$(\beta-1)U\frac{\partial\rho}{\partial x} + (\beta-1)\rho\frac{\partial U}{\partial x} + \frac{1}{2}\rho U\frac{\partial\beta}{\partial x} = 0 \tag{6.1}$$

This result has important implications because it can be interpreted as placing constraints on possible values of $\partial\beta/\partial x$. For this purpose, it is convenient to re-present the equation in the following form:

Continuity (implications for $\partial\beta/\partial x$):

$$\frac{1}{2}\rho U\frac{\partial\beta}{\partial x} = -(\beta-1)U\frac{\partial\rho}{\partial x} - (\beta-1)\rho\frac{\partial U}{\partial x} \tag{6.2}$$

This equation can be used to eliminate $\partial\beta/\partial x$ from any of the equations in Sections 4 & 5. An especially useful application of this capability is in the area-weighted momentum equation (Eq. (4.9)), giving:

Momentum (enhanced area formulation):

$$\frac{\partial p}{\partial x} - (\beta-1)U^2\frac{\partial\rho}{\partial x} + \rho\frac{\partial U}{\partial t} + \rho U\frac{\partial U}{\partial x} = \frac{F_{\tau,w}}{A} \tag{6.3}$$

which is a uniaxial momentum equation that does not include α and that does not include any derivative of either β or α. If the speed of sound c satisfies

$$c^2 = \frac{dp}{d\rho} \tag{6.4}$$

Eq. (6.3) can be simplified further to

Momentum (enhanced area formulation):

$$\left[1 - (\beta-1)\frac{U^2}{c^2}\right]\frac{\partial p}{\partial x} + \rho\frac{\partial U}{\partial t} + \rho U\frac{\partial U}{\partial x} = \frac{F_{\tau,w}}{A} \tag{6.5}$$

Eq. (6.5) is the one that is referred to in the Abstract and in the Introduction. Although it is valid for uniaxial flows with non-uniform velocity distributions, it does not include any derivative of β or α and it does not include α itself. Furthermore, the coefficient of $\partial p/\partial x$ can differ significantly from unity only when at least one of $(\beta-1)$ and U/c is not small. This situation will rarely arise. Firstly, large values of β imply exceptionally large deviations from uniform velocity distributions that are much more likely to be associated with flow reversals than with flows in which all velocity vectors in a cross-section are in the same direction. If so, it is reasonable to expect that the *mean* velocity U will be relatively small. Secondly, even if large values of β were to occur in other cases, their influence would still be weighted by the square of the Mach number ($M = U/c$). In the case of liquid flows, this is usually very small indeed – e.g., $M^2 < 10^{-5}$ in typical flows of water in pipelines.

6.1. Re-evaluation of the plug flow form of the momentum equation

Except for this small difference in the coefficients of $\partial p/\partial x$, Eq. (6.5) is identical to the simple plug flow Eq. (3.2). An almost inevitable conclusion from this result is that the widespread practice of neglecting the influence of β, α, and their derivatives in simulations of transient liquid flows can be justified rigorously. The only remaining doubt is how the product of $(\beta-1)$ and $(U/c)^2$ behaves when β tends to infinity and U tends to zero. For the particular case of a step change such as that caused by a sudden wavefront, this doubt is itself all–but removed in Appendix-1, in which it is shown that the product remains almost constant across the step.

6.2. Re-evaluation of the area-weighted form of the uniaxial momentum equation

Eq. (6.5) can also be used to assess the suitability of the area-weighted momentum Eq. (4.9) when, as is common, the term involving the derivative $\partial\beta/\partial x$ is neglected. With this simplification, the area-weighted equation differs in two respects from the more correct Eq. (6.5). Firstly, the coefficient of $\partial p/\partial x$ is unity whereas it should be smaller than unity, albeit usually only very slightly so. Secondly, the term $\rho U\partial U/\partial x$ is multiplied by β, which is always greater than one. Both of these changes have the effect of changing the relative importance of the respective terms in comparison with the inertial term $\rho\partial U/\partial t$. However, the proportional error in the coefficient of $\partial p/\partial x$ will usually be much smaller than that in the coefficient of $\rho U\partial U/\partial x$. When this is the case, the inevitable conclusion must be that the most commonly used method of attempting to allow for the consequences of velocity distributions does not improve solutions. On the contrary, it actually downgrades them. In the interests of transparency, it must be admitted that the authors have themselves been caught in this trap for decades.

For completeness, it is worth noting that, although the primary focus of this paper is on wave-like flows, the analysis presented thus far is valid for steady flows as well as for unsteady flows (simply set $\partial U/\partial t = 0$ in the equations). Although such simulations are often undertaken in a manner that neglects all density changes, thereby greatly simplifying the analysis, the true flow cannot be exactly incompressible, and the more complete equations presented above can allow for consequences of this.

6.3. Re-evaluation of the flow-weighted form of the uniaxial momentum equation

The full flow-weighted Eq. (5.4) differs so much from Eq. (6.5) that little benefit can be gained from a detailed comparison. However, it is noted in passing that Eq. (5.4) includes a temporal derivative $\partial\beta/\partial t$ as well as a spatial derivative $\partial\alpha/\partial x$. The explicit inclusion of either of these terms in an analytical or numerical model would almost certainly be prohibitive. However, it is even more difficult to estimate them than $\partial\beta/\partial x$ and yet their influence seems likely to be stronger. Furthermore, even if they are simply neglected, the resulting simplified form of Eq. (5.4) will differ from Eq. (6.5) in the following ways:

(i) the coefficient of $\partial p/\partial$x is unity,
(ii) the term $\rho\partial U/\partial t$ is multiplied by β,
(iii) the term $\rho U\partial U/\partial x$ is multiplied by α,
(iv) the source term on the right hand is expressed differently.

Once again, the relative importance of the various terms is altered, this time more strongly than in the area-weighted case. However, it is more difficult to assess the consequences of this because the right-hand side term is different. Nevertheless, it is noted that the *relative* importance of the $\rho\partial U/\partial t$ and $\rho U\partial U/\partial x$ terms is changed by a factor of α/β, which will usually be greater that the corresponding factor β in the area-weighted case. These disadvantages are in addition to the complications arising in the evaluation of the right-hand side term when the mean velocity is close to zero.

7. Relationship to previous work

A brief review of previous work is now given, with special emphasis on the relation to the momentum Eqs. (4.9) & (5.4). Almeida and Koelle ([9], pp. 29–31) presented both equations (from Almeida's Ph.D. Thesis [13]). By eliminating the pressure gradient (i.e., subtracting Eq. (5.4) from Eq. (4.9)) they obtained an expression for the unsteady wall shear that included the influence of so-called 'virtual inertia' effects to allow for influences of velocity distributions that, herein, are retained on the left-hand sides of all equations. Subsequently, Abreu and Almeida [11,12,14] employed a hybrid 1D / 2D model to confirm their proposed use of their expression and to obtain instantaneous values of β and its time derivative.

Brunone and Golia [15] used the Boussinesq coefficient β (which they referred to as a Coriolis factor) as a measure of the non-uniformity of unsteady cross-sectional velocity distributions. They presented three time-plots of $\beta \sim t$ for the oscillatory-flow measurements of Hino et al. [16]. One such figure is reproduced here as Fig. 3 (reprinted by permission from Springer). Over most of the cycle, the absolute value of β is less than, say, 1.4. However, much larger values pertain during periods when the mean flow is close to zero. In such periods, the time-derivative of β, which appears in Eq. (5.4), can be orders of magnitude larger than unity.

An influential paper by Brunone et al. [10] has been one inspiration for the current work. It presents the area-weighted and flow-weighted methods and explores the consequences of the different meanings of the source terms in Eqs. (4.9) and (5.4). To validate their approach, they used a 2-D model [17] which has co-axial cylindrical elements, each of which can be sufficiently thin for the axial velocity to be regarded as radially uniform. This avoids the need for consideration of values of the Boussinesq and Coriolis coefficients or their derivatives within individual elements, even though the overall solution provides time-dependant velocity profiles that can be used to infer cross-sectional values of β and α and their derivatives. For the case of laminar flow, they used this information as prescribed input to area-weighted and flow-weighted 1-D models and found very close agreement with the 2-D results in both cases.

In a study of open-channel flow, Xia and Yen [18] numerically simulated the propagation of one positive phase of a sinusoidal surface wave in a hypothetical test case in an 87 km long channel with an initially steady flow. They considered a

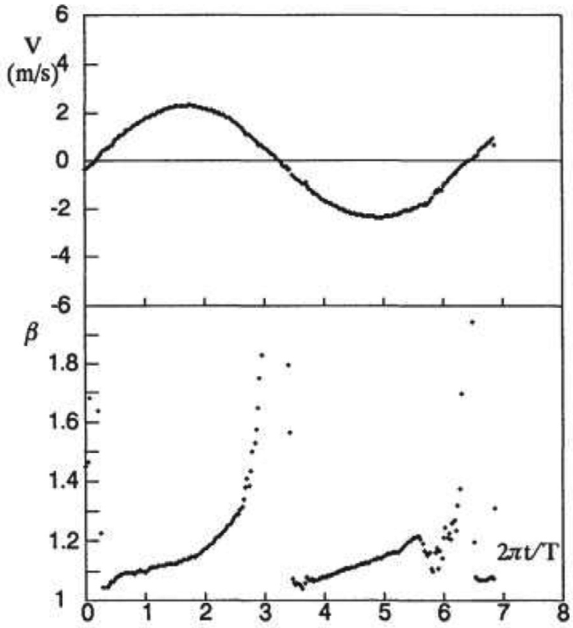


Fig. 3. Time histories of mean velocity and β in Hino's test no. 9. Copied from Brunone and Golia [15].

range of cases including both spatially uniform and spatially non-uniform values of β. Restricting attention to cases where they did not simultaneously vary one or more pressure-correction coefficients, the maximum differences between solutions obtained with constant values of $\beta(x,t) = 1$, 1.33 & 2 were small (rarely as much as 1%). Other simulations included prescribed constant spatial variations in β, but these results are not cited herein because two pressure correction coefficients were varied simultaneously and the authors concluded that their influence dominated that of β. Field et al. [19] considered unsteady flow in open channels with depth-dependant Boussinesq and Coriolis coefficients for velocity distributions over compound cross-sections. They reported significant differences between area- and flow-weighted formulations.

8. Method of characteristics (MOC)

Attention now turns to the integration of the partial differential equations in space and time, in particular by the Method of Characteristics (MOC), which is widely used in academia and in professional practice. In essence, one continuity equation and one momentum equation are combined linearly in a form that can be expressed in ordinary differential form. Usually (as herein) some development of the basic equations is undertaken before they are combined and this influences the resulting equations, but the same basic methodology is applicable in all cases. In this Section, the plug-flow formulation is presented first and then one uniaxial form is presented.

8.1. Plug flow formulation

For the particular case of the plug flow formulation, Eqs. (3.1) & (3.2) are combined, using Eq. (6.4), to give the compatibility equations:

MOC equations (plug flow):

$$\lambda \frac{dp}{dt} + \rho c \frac{dU}{dt} = c \frac{F_{\tau,w}}{A} \tag{8.1}$$

which are valid (only) in the particular directions:

MOC directions (plug flow):

$$\frac{dx}{dt} = U + \lambda c \tag{8.2}$$

where λ is either of

$$\lambda = \pm 1 \tag{8.3}$$

The parameter λ is introduced here to simplify comparisons with equations presented below for area-weighted formulations.

It is worth noting in passing that exactly the same compatibility equations (i.e., Eq. (8.1)) are obtained if Eqs. (3.1) & (3.2) are approximated by neglecting $U\,\partial/\partial x$ terms in comparison with $\partial/\partial t$ terms before the combination is performed. The only difference in the resulting MOC formulations is that the compatibility equations are integrated in the directions $dx/dt = \pm c$ instead of in the true directions $U \pm c$. This can simplify numerical integrations of the equations, but it is acceptable only when the Mach number $|U/c| << 1$. One disadvantage of this approximation is that it renders MOC methods incapable of reproducing steady flow conditions *exactly*. Another is that it makes them inherently unsuitable for simulating any phenomenon in which wavefront-steepening effects are important. Indeed, when it is additionally assumed that the sound speed c is independent of x & t, no inertial steepening at all is predicted. Fortunately, these matters are of negligible importance in many practical applications of MOC, so it is commonly sensible to take advantage of the numerical simplifications that they enable. Nevertheless, they can be important when the Mach number is not negligible in comparison with unity or when studying effects such as unsteady friction that influence wavefront steepening (e.g. [20,21]).

8.2. Enhanced area-weighted formulation

The corresponding MOC equations for the enhanced area-weighted, uniaxial case based on Eqs. (4.7) and (6.5) are:

MOC equations (enhanced area-weighted):

$$\lambda\frac{dp}{dt} + \rho c\frac{dU}{dt} = c\ \frac{F_{\tau,w}}{A} \tag{8.4}$$

which are valid (only) in the particular directions:

MOC directions (enhanced area-weighted):

$$\frac{dx}{dt} = U + \lambda c \tag{8.5}$$

where λ is either of

$$\lambda = \pm\sqrt{\left[1 - (\beta - 1)\frac{U^2}{c^2}\right]} \tag{8.6}$$

These make full allowance for variations in β. The expression in square brackets in Eq. (8.6) is the same as the coefficient of $\partial p/\partial x$ in the corresponding momentum equation (Eq. (6.5)) and, as indicated in the discussion presented after that equation, it will rarely differ strongly from unity in flows of practical interest to hydraulic engineers. In a nutshell, therefore, it will be rare for significant advantage to accrue from allowing for non-uniform velocity distributions in simulations of strong transients in uniform-area pipes. This deduction is illustrated quantitatively in Appendix-2.

8.3. Conventional area-weighted formulation

The MOC equations for the conventional area-weighted, uniaxial case based on Eqs. (4.7) and (4.9) are:

MOC equations (enhanced area-weighted):

$$\lambda\frac{dp}{dt} + \rho c\frac{dU}{dt} = c\frac{F_{\tau,w}}{A} - \frac{1}{2}\rho c U^2\frac{\partial\beta}{\partial x} \tag{8.7}$$

which are valid (only) in the particular directions:

MOC directions (enhanced area-weighted):

$$\frac{dx}{dt} = \beta U + \lambda c \tag{8.8}$$

where λ is either of

$$\lambda = -\frac{1}{2}(\beta - 1)\frac{U}{c} \pm \sqrt{\left[1 + \frac{1}{4}(\beta - 1)^2\frac{U^2}{c^2}\right]} \tag{8.9}$$

Appendix-2 includes comparisons between these equations and the enhanced area-weighted counterparts. These confirm the conclusions expressed above about the consequences of neglecting the influence of $\partial\beta/\partial x$.

9. Flow regimes and absolute values of β and α

The analysis presented above removes the need for $\partial\beta/\partial x$ to be considered explicitly. Furthermore, except in extreme cases in which it is not reasonable to assume $(\beta-1)M^2 << 1$, it effectively also removes the need for β itself to be considered explicitly. That is, the influence of the velocity distribution and changes thereof is implicitly incorporated in the wall friction term $F_{\tau,w}$. To the authors' knowledge, this has never been proved previously even though it is implicitly assumed in much work on the development of expressions for unsteady skin friction. Furthermore, it gives strong support to Brunone et al.'s [10] conclusion that the skin-friction term required in plug-flow formulations of the equations depends solely on stresses at the wall itself – although strictly, as shown herein, this statement applies to the enhanced-area formulation.

Notwithstanding this achievement, because the present paper deals solely with 1-D methods, it provides no information whatsoever on actual velocity distributions or on the resulting wall shear forces that need to be input to practical analyses. The authors are not aware of any plausible method of relating β to instantaneous values of uniaxial flow parameters (U, $\partial U/\partial t$ and $\partial U/\partial x$) that may be readily available during 1-D simulations. However, Kruisbrink and Tijsseling [22] have proposed a transport equation for β for use in the recovery period following a sudden change in mean velocity, noting that the asymptotic value of β after recovery will normally correspond to steady flow. Their method implicitly assumes quasi-steady conditions to prevail before the sudden change. Here, it is sufficient to observe that two very different approaches are typically followed in the development of models of unsteady friction. A review of both methods is given by Bergant et al. [23]. One, dealing with short-lived transient events, uses convolution methods to estimate the influence of very rapid ('transient') flow changes in which the wall shear is dominated by short-term adjustments of the flow in response to localised pressure changes typical of wavefronts. The other approach is best suited to much longer timescales in which the flow resembles more closely a quasi-uniform acceleration in which the flow profile changes in an almost quasi-steady manner. That is, the flow accelerates, but the *shape* of the velocity profile is almost spatially uniform. Unfortunately, in typical practical unsteady flows involving pressure transients, both of these flow behaviours exist because relatively long periods exist between periods of strong transient action.

From a purely theoretical perspective, the flow-weighted approach presented above is quite revealing because the coefficient of the inertial acceleration $\partial U/\partial t$ includes the factor β. Consider, for instance, a gradually-accelerating, laminar flow in a pipe of circular cross-section. For this, β will be approximately 4/3. Schönfeld's [24] seminal paper discloses "apparent inertia" from analytical solutions of axisymmetric, incompressible, laminar flow and relates it to the Boussinesq coefficient of steady flow. Young and Tsai [25] show that the value of 4/3 is valid for low frequencies but that the value 1 prevails for high frequencies, noting here that the factor of 4/3 is needed for a starting flow induced by a suddenly-imposed pressure rise [26]. Young and Tsai also introduce a frequency-dependant factor to correct linear resistance at high frequencies.

Bessems et al. [27] introduce correction coefficients for the pressure gradient and the resistance based on a two-layer model for velocity profiles. The latter depend on both the mean velocity and the pressure gradient. They do not explicitly introduce β, but use the average value of u^2 which, according to Eq. (1.2), is equal to $\beta\ U^2$. In their Appendix A, they give a nonlinear expression for $\beta\ AU^2$ in terms of U and $\partial p/\partial x$.

Achard and Lespinard [28] and Letelier and Leutheusser [29] consider the higher-order derivatives ignored by Schönfeld [24] and make the wall shear stress proportional to U, $\mathrm{d}U/\mathrm{d}t$ and $\mathrm{d}^2U/\mathrm{d}t^2$. This can be regarded as a marginal extension of Brunone-type unsteady friction modelling. Letelier and Leutheusser present analytical and experimental results for starting flow caused by sudden valve opening.

10. Practical applications

It is appropriate to consider the implications of the theoretical development presented above to engineering practice. For this purpose, consideration is given to two practical applications of unsteady flows with strongly differing characters.

10.1. Water distribution networks

The first application is the study of water-hammer events in water distribution networks. Typically, these are caused by rapid valve closures or rapid pump start-up. Such events cause sudden changes that, as a first approximation, are commonly imagined as step wavefronts even though they are actually ramp wavefronts with lengths of several (usually many) pipe diameters. These wavefronts travel along the network, responding gradually and cumulatively to wall friction and responding rapidly to encounters with pipe branches and air valves, etc. A pressure or velocity sensor at any particular location would record relatively abrupt changes when wavefronts pass and much slower changes during long intervals between wavefronts. To simplify the following discussion, the flow between the sudden events is somewhat loosely termed 'quasi-steady'.

Any accurate method of simulating such phenomena must be able to handle both types of flow – i.e., rapidly-changing and quasi-steady. In general, it must also be able to handle phenomena such as cavitation and column separation, although additional complications such as these are excluded from the present discussion. For analysts, one major advantage of liquid flows in pipes is that the Mach number is usually very small, typically much less than ½%. Another advantage, less often noted explicitly, is that the purpose of the simulations is usually to predict amplitudes, not rates-of-change of amplitude. These factors justify the omission of convective terms from the governing Eqs. (3.1) and (3.2), leading to the simplification described in Section 8.1. Nevertheless, any analysis that is intended to handle effects that depend on the Boussinesq coefficient needs to be able to deal with (i) step-like changes in β at wavefronts, (ii) less rapid variations in β in extended regions of flow behind wavefronts and (iii) subsequent regions of almost quasi-steady flow. Furthermore, the wall-shear stress will also vary rapidly, less rapidly and gently in these regions. This poses big challenges for analysts, but the task will be less-daunting when there is no need for explicit consideration of derivatives of the Boussinesq and Coriolis coefficients.

10.2. Wavefront steepening in gas flows

The second application is the simulation of steep-fronted waves in unsteady gas flows in ducts. Wavefront steepening can lead to the development of shocks [30] and it can influence the frequencies of sound emitted from ducts such as

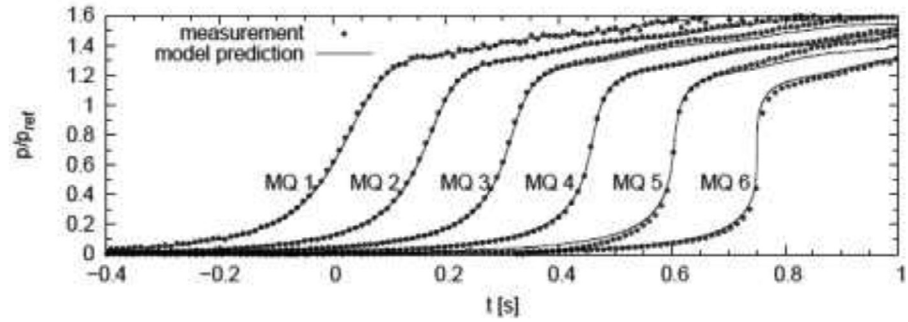


Fig. 4. Wavefront steepening in a railway tunnel (pressures at successive locations) by Adami and Kaltenbach [20].

vehicle exhaust systems and railway tunnels [20,21]. Fig. 4 (reproduced with kind permission of Deutsche Bahn AG, DB Systemtechnik) shows that the steepening effect can be strong even at low Mach numbers. As a simple example, consider a single, compressive wavefront propagating into an initially steady flow (1) and causing a rapid change to a new quasi-steady state (2). The leading toe of the wavefront will travel at a speed $c_1 + U_1$ whereas its heel will travel at a speed $c_2 + U_2$. In practice, $c_2 > c_1$ and $U_2 > U_1$ so the heel will travel faster than the toe, thereby causing the wavefront to shorten continuously. In a sufficiently long duct, it will eventually evolve into a shock unless phenomena that resist the shortening are present.

In studies of such wavefronts, it is essential to retain the convective terms in the governing equations because the rate of shortening depends more strongly on these than on the increase in the speed of sound. If a uniaxial representation is required, the enhanced area-weighted Eq. (6.5) is especially valuable because the other formulations involve derivatives that could not be evaluated reliably even if quasi-steady values of β and α could be estimated with confidence.

11. Closing remarks

The influence of non-uniform velocity distributions on various uniaxial formulations of continuity and momentum equations has been assessed for unsteady flows in ducts of uniform area. Special attention has been paid to the occurrence of the Boussinesq and Coriolis coefficients β and α. It has been shown that the practice of disregarding *derivatives* of these coefficients in the fundamental equations (whilst retaining β and α themselves) is less satisfactory than disregarding the influence of the coefficients altogether.

The method used to prove this outcome has utilised both area-weighted and flow-weighted methods of integrating the basic equations over the flow cross-section. It has been shown that a combination of the two methods can be used to enable the derivative $\partial\beta/\partial x$ to be replaced by a function of the derivatives of velocity and density. The elimination of $\partial\beta/\partial x$ in this way has led to an enhanced area-weighted formulation of the momentum equation that, in almost all flows of practical relevance to hydraulic engineers, differs negligibly from the corresponding equation obtained when no account whatsoever is taken of β and α. That is, the adoption of this practice, which has previously been justified only on pragmatic grounds, now has a formal theoretical justification.

The only difference between the enhanced area-weighted momentum equation and its simple plug flow counterpart is the coefficient of $\partial p/\partial x$. This difference is approximately proportional to $(\beta-1)U^2$, which is shown in the Appendix to be almost invariant during the passage of a wavefront. This property can be used to estimate the change in β caused by the passage of the wavefront.

It has been shown that the benefits of avoiding the need for the inclusion of derivatives of β and α in the momentum equation are especially strong in simulations based on the Method of Characteristics – because the derivatives influence the characteristic directions as well as the compatibility equations.

As a related issue, attention has also been given to other consequences of alternative methods of weighing the uniaxial equations to allow for cross-sectional variation in velocity. One outcome is a demonstration that these have different consequences for the interpretation of sources terms. In the particular case of the term describing the influence of fluid shear, some formulations require direct knowledge of the distribution of shear over the whole of the flow cross-section, whereas others require knowledge only of the shear force on the pipe wall.

Although the principal purpose of the paper relates to flows in which pressure transients exist, the equations can also be applied to steady flows by simply setting all $\partial/\partial t$ terms to zero. One implication of this is that, strictly, the coefficient of $\partial p/\partial x$ in the enhanced area-weighted momentum equation applies in steady flows as well as unsteady flows.

For the avoidance of doubt, it is emphasized that no attempt has been made to provide formulae enabling actual values of the coefficient β itself to be determined. This cannot be done reliably using only cross-sectional averaged values of flow parameters.

Also, for the avoidance of doubt, it is repeated that the analysis is *not* directly equivalent to analyses of so-called rigid column accelerations, in which all changes in density are neglected and so, in effect, the wave speed is infinite. In contrast, the analysis is applicable to so-called 'weakly compressible' water-hammer phenomena in which the density is pressure-dependant. In such analyses, convective terms are usually neglected in the governing equations and, in this case, β has a role only in the flow-weighted formulations of the equations.

Data Availability

Data will be made available on request.

Appendix-1. Invariance of the product $(\beta-1)\rho^2U^2$

A reviewer of the first version of this paper has provided the authors with a truly elegant derivation of a useful result that generalises a result previously presented by Kruisbrink and Tijsseling [22]. The new derivation is especially valuable because it follows directly from Eq. (6.2) that underpins the method of eliminating the partial derivative of β in the derivation of the enhanced area-weighted analysis. Reproduced with the reviewer's permission, the new derivation is as follows:

Expression for $\partial\beta/\partial x$ derived in Section 6:

$$\frac{1}{2}\rho U\frac{\partial\beta}{\partial x} = -(\beta-1)U\frac{\partial\rho}{\partial x} - (\beta-1)\rho\frac{\partial U}{\partial x} \tag{A1.1}$$

This relationship can be rearranged as

$$\frac{1}{\beta-1}\frac{\partial\beta}{\partial x} = -2\left(\frac{1}{\rho}\frac{\partial\rho}{\partial x} - \frac{1}{U}\frac{\partial U}{\partial x}\right) \tag{A1.2}$$

or

$$\frac{1}{\beta-1}\frac{\partial(\beta-1)}{\partial x} = -\frac{2}{\rho U}\frac{\partial(\rho U)}{\partial x} \tag{A1.3}$$

On integration, this gives

$$\ln(\beta-1) = -2\,\ln(\rho U) + f\{t\} \tag{A1.4}$$

and so

$$(\beta-1)\rho^2U^2 = \mathrm{e}^{f\{t\}} \tag{A1.5}$$

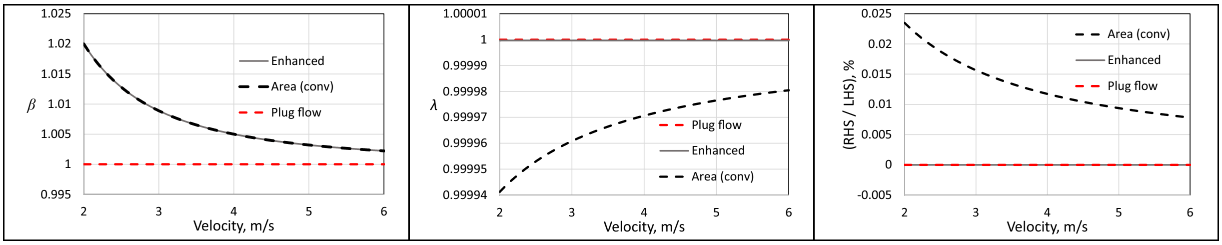
At first sight, this result can seem counter-intuitive. It is an unequivocal statement that, at any particular time t, the product $(\beta-1)\rho^2U^2$ is uniform along the duct or pipe. Moreover, this is true even though no assumption has been made about axial variations of flow along the duct. That is, it is true regardless of the presence of waves, and it is shown in the following paragraph that it follows that the product is also independent of time. This Appendix then closes with a detailed consideration of any possible reasons why the outcome could be false.

Consider an initially steady flow along the pipe. In this case, the product ρU is exactly uniform along the pipe and, even without using the above derivation, it would be expected that the product $(\beta-1)\rho^2U^2$ is either uniform or very nearly so. Now consider a disturbance initiated at an upstream boundary and propagating downstream. At any location, changes caused by the disturbance must satisfy Eq. (A1.5), but so must the behaviour at downstream locations that the disturbance has not yet reached. These two requirements can be satisfied simultaneously only if the product $(\beta-1)\rho^2U^2$ is independent of time as well as distance. That is, waves do not, by themselves, cause changes in the value of the product.

This outcome has been deduced solely from consideration of relationships obtained from considerations of continuity – i.e. Eqs. (4.7) and (5.3). It is necessarily valid within the limits of the assumptions made in the derivation of those equations. No assumptions have been made about the variation of axial velocity, either along the duct of over its cross-section. Also, the Boussinesq coefficient is defined in terms of velocity and area alone so it, too, cannot limit the validity of the conclusion. The only remaining possible cause of error is the assumption that cross-sectional variations in density are negligible. It is certainly the case that the cross-sectional variations will exist, but it is also true that they will be much smaller than axial variations associated with pressure changes caused by waves. Accordingly, it is concluded that the product $(\beta-1)\rho^2U^2$ is indeed very nearly constant in time and uniform in space. Use is made of this property in Appendix-2.

Appendix-2. Influence of β on the accuracy of MOC integrations

It has been shown in the main text that the plug flow approximation introduces smaller errors than those associated with the conventional method of attempting to allow for the influence of non-uniform velocity distributions in flow cross-sections. This has been done by comparing two conventional analyses with a new, nominally exact, analysis. So far, however, the numerical differences between solutions obtained using the three methods have not been quantified. In principle, one way to do this would be by undertaking simulations of specific flow cases. Unfortunately, this approach would be far from ideal in the present context, partly because some of the differences are very small, but also because realistic simulations



(a) β - Eq.A1-5 with constant $f\{t\}$ (b) λ – Eqs.8.3, 8.6 & 8.9 (c) Error when neglecting $\partial\beta/\partial x$

Fig. A2–1. Acceleration from 2 m/s to 6 m/s; nominally turbulent velocity distribution.

would involve additional complications such as friction and valve closures that would obscure the target effects. The following approach avoids such complications and enables a wide range of cases to be explored. For clarity, inviscid conditions are assumed, thereby enabling attention to focus exclusively on the influence of velocity distributions.

Each figure in this Appendix shows variations of β and λ for three analyses. Graphs labelled 'Enhanced' show the enhanced area-weighted solution (i.e. the analytically correct solution) and the other two graphs in each box show the corresponding plug-flow approximation and the conventional area-weighted approximation. The figures also show the magnitude of the term in $\partial\beta/\partial x$ on the right-hand side of Eq. (8.7), expressed as a proportion of the sum of the corresponding left-hand-side terms. Only the equations corresponding to $dx/dt = U + c$ are shown.

The initial condition, which is prescribed by a mean flow velocity and a nominal value of β, is representative of a steady flow. The flow is then disturbed by a ramp-fronted wave that is deemed to accelerate (or decelerate) the flow at a constant rate to a new prescribed mean velocity. The solutions have been obtained by integrating along the characteristic line in 1000 steps, using finite differences and have been shown to differ negligibly from the corresponding solutions using only 100 steps. The true (enhanced-area-weighted) solution is obtained first, giving the correct variations of β, p and ρ during the acceleration together with the variation of λ corresponding to Eq. (8.6). The prescribed velocity and the inferred values of β are then used as input parameters to the conventional-area-weighted analysis, giving its implied values of pressure and density and the variation of λ corresponding to Eq. (8.9). Finally, the ratio of the right-hand-side and left-hand-side of Eq. (8.7) is evaluated. The figures compare the outcomes of the two analyses and also that obtained using the simple approximation.

The chosen initial values of β in the area-weighted formulations, namely 4/3 and 1.02, are representative of laminar and turbulent flow respectively. This is useful even though no account is taken of viscosity in the integrations. This is because, even in real flows, time-scales corresponding to vorticity diffusion are much greater than those corresponding to inertial waves. As a consequence, the dominant cause of variations in β during the passage of a sudden wave is the induced change in the mean velocity, not the small, vorticity-induced changes in the cross-sectional variation of local velocities.

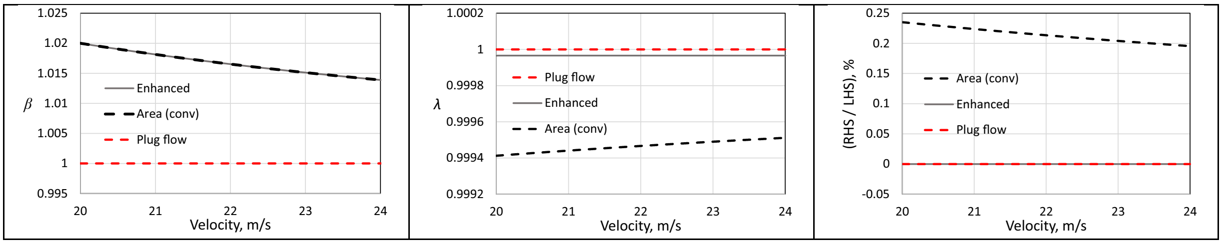
In all of the following examples, the chosen fluid, namely air, is treated as a perfect gas for which the gas constant is 287 J/kg.K and the ratio of the principal specific heats is 1.4. The initial pressure and temperature are 100 kPa and 15 °C respectively, implying an initial density and speed of sound of approximately 1.209 kg/m^3 and 340.2 m/s respectively.

In Fig. A2-1, the initial value of β in the area-weighted cases is 1.02 and this reduces as the air flow accelerates from 2 m/s to 6 m/s. The value of β necessarily reduces as the velocity increases during the acceleration because the local changes in density and velocity are deemed to be uniform over the whole cross-section. This would not be exactly true in a real flow because the no-slip condition at the wall would cause near-wall conditions to deviate from the idealised state and consequential vorticity diffusion would immediately influence conditions very close to the wall. Nevertheless, as indicated above, the assumed behaviour will be a close approximation to reality for rapid changes encountered during the passage of a sudden wavefront.

The values of λ implied by both area-weighted methods are close to unity. However, it is noteworthy that the simple plug-flow approximation (which implies $\lambda = 1$) is a much closer approximation to the true ('enhanced') value than is the conventional area-weighted solution. This outcome is also seen in each of the following examples.

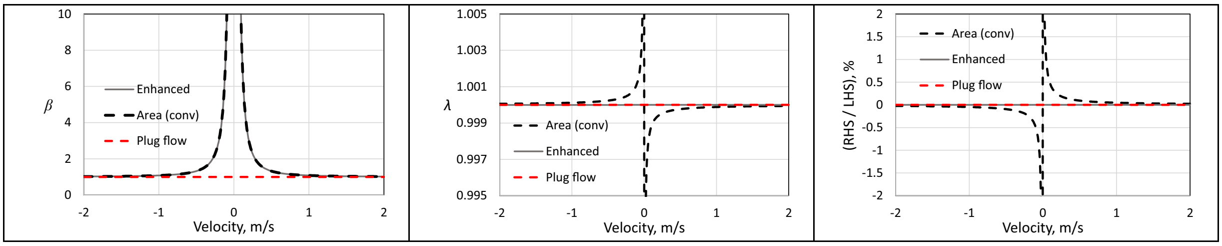
The right-hand side term is non-zero only for the conventional area-weighted method. When expressed as a proportion of the left-hand side terms, there is no dependence on the assumed length of the wavefront because the ratio $\Delta x/\Delta t$ is constrained to be equal to $U + c$. In this particular example, the ratio RHS/LHS is small and this illustrates why the usual practice of discarding such terms in practical analyses has not led to obvious inconsistencies when analysing low Mach-number flows. However, the following examples include cases where this practice would be less satisfactory.

Fig. A2-2 shows corresponding outcomes for an equal change in velocity, namely 4 m/s, but starting from a velocity that is ten times greater. Again, the plug flow approximation performs far better than the conventional area-weighted formulation. For both of them, the errors are much greater than those shown in Fig. A2.1 for smaller average velocities, but the absolute errors are nevertheless still small. Indeed, this conclusion remains true even if the initial rate of flow is increased by a further factor of ten. In that case (not shown), the partial derivative term in the conventional area-weighted method is less than 2.5% of the sum of the left-hand-side terms.



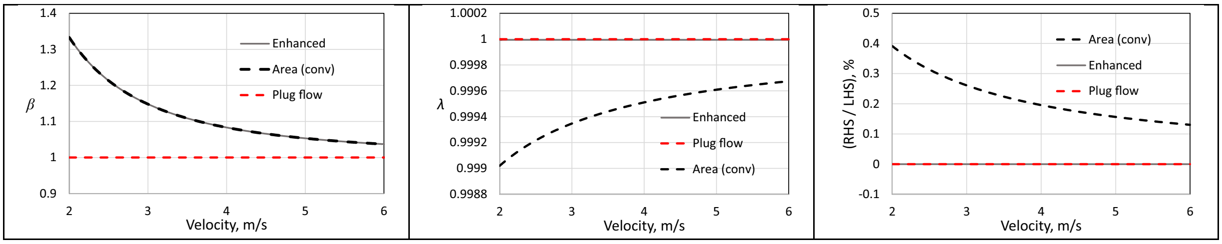
(a) β - Eq.A1-5 with constant $f\{t\}$ (b) λ – Eqs.8.3, 8.6 & 8.9 (c) Error when neglecting $\partial\beta/\partial x$

Fig. A2–2. Acceleration from 20 m/s to 24 m/s; nominally turbulent velocity distribution.



(a) β - Eq.A1-5 with constant $f\{t\}$ (b) λ – Eqs.8.3, 8.6 & 8.9 (c) Error when neglecting $\partial\beta/\partial x$

Fig. A2–3. Deceleration from +2 m/s to −2 m/s; nominally turbulent velocity distribution. (*All vertical axes are curtailed to highlight behaviour as the flow reverses*)



(a) β - Eq.A1-5 with constant $f\{t\}$ (b) λ – Eqs.8.3, 8.6 & 8.9 (c) Error when neglecting $\partial\beta/\partial x$

Fig. A2–4. Acceleration from 2 m/s to 6 m/s; nominally laminar velocity distribution.

A very different outcome is obtained when the acceleration or deceleration causes a reversal of the mean flow direction. This is illustrated in Fig. A2-3, for which the change in mean velocity is again 4 m/s, but the initial velocity is −2 m/s. When any flow with a non-uniform velocity distribution passes through a state of zero mean velocity, the Boussinesq and Coriolis coefficients become infinite. This can be handled analytically, but it can pose major problems for numerical analysts. This issue does not arise for the enhanced area-weighted solution because β appears in Eq. (8.3) only in the product $(\beta-1)M^2$, which has been shown in Appendix A1 to be almost constant. However, this advantage does not exist for the conventional area-weighted method because β appears in the product $(\beta-1)M$ as well as in $(\beta-1)M^2$. Furthermore, as seen in Fig. A2-3, the *sign* of λ changes when the mean velocity crosses zero, as also does the sign of the contribution of β to the right-hand-side term in Eq. (8.8).

The final example in this Appendix is a repeat of the acceleration from 2 to 6 m/s, but assuming an initial pseudo-laminar velocity profile for which $\beta = 4/3$. Fig. A2-4 shows that the outcome is visually similar to that for the corresponding case of turbulent flow shown in Fig. A2-1 except that the magnitudes of the maximum values of $(1-\lambda)$ and of the RHS/LHS ratio are greatly increased – by factors of approximately 16, which is approximately equal to the ratio of the values of $(\beta-1)$ for the nominally laminar and turbulent profiles.

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