Numerical data related to the Lagarias-Soundararajan xyz-conjecture

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For positive coprime integers X, Y, Z with $X + Y = Z$ we define the *quality function*

$$Q^*(X, Y, Z) = \frac{3}{2} \frac{\log \log Z}{\log P(XYZ)},$$

where $P(x)$ is defined as the largest prime factor of the integer x.

Lagarias and Soundararajan [LS] conjecture, based on heuristic arguments, that

$$\limsup_{Z \to \infty} Q^*(X, Y, Z) = 1.$$

They call this the *strong form* of the xyz-*conjecture*, the weak form being that this lim sup exists as a finite positive number.

In this note I gather some numerical data on $Q^*(X, Y, Z)$. As far as I know the largest available database of interesting triples X, Y, Z is provided by the "ABC@home" project [ABC]. The aim of this project is to find interesting triples of coprime X, Y, Z with $X + Y = Z$ for which a different quality function is large, namely

$$q(X, Y, Z) = \frac{\log Z}{\log R(XYZ)},$$

where $R(x)$ is the radical of the integer x, defined as the product of its distinct prime factors. To be precise, the "ABC@home" project aims at computing the complete set of triples with $Z < 10^{18}$ and $q(X, Y, Z) > 1$, and actually finds also many triples with $10^{18} \leq Z < 2^{63}$ and $q(X, Y, Z) > 1$. This gigantic effort is expected to finish soon. At May 8, 2012 the database contained 23 076 546 triples. I computed $Q^*(X, Y, Z)$ for these triples, and selected those with $Q^*(X, Y, Z) > 1.3$. There are 1737 of them, of which the 137 with $Q^*(X, Y, Z) > 1.4$ are presented in Table 1 below, sorted for $Q^*(X, Y, Z)$. All 1737 triples can be found in http://www.win.tue.nl/~bdeweger/downloads/xyz1737.txt.

Table 1: Triples with $Q^*(X, Y, Z) > 1.4$

X	Y	Z	P(XYZ)	q(X, Y, Z)	Q*(X, Y, Z)
1	4374	4375	7	1.56789	1.63904
1	2400	2401	7	1.45567	1.58180
91	1771470	1771561	13	1.39548	1.55903
4026275	1371299293	1375325568	19	1.30796	1.55200
3466384181	162615350840320	162618817224501	31	1.25738	1.52362
2197	583443	585640	13	1.28812	1.51248
78125	27217619	27295744	17	1.42158	1.50379
5679428	196088607	201768035	19	1.18866	1.50328
5	1024	1029	7	1.29721	1.49296
69984	18458141	18528125	17	1.27327	1.49168
1446484375	4867359029	6313843404	23	1.1739	1.49089
2118662369117	11584597005883	13703259375000	31	1.16232	1.48928
1771561	94712814	96484375	19	1.1428	1.48323
25154560	3184461423	3209615983	23	1.1387	1.47632
121	255879	256000	13	1.48887	1.47486
17332693	2811467307	2828800000	23	1.13213	1.47356
2116453031	4292937252864	4295053705895	31	1.25429	1.4722
3	125	128	5	1.42657	1.472
176202799695875	3178472661789594624	3178648864589290499	47	1.13327	1.46173
289	42471000	42471289	19	1.09179	1.45997
1127357	5688387	6815744	17	1.36424	1.45905
131769	289478257991	289478389760	29	1.16826	1.45801
12005	161051	173056	13	1.3094	1.45617
390625	35995648	36386273	19	1.1615	1.45547
1169745467950813	6231375784189187	7401121252140000	41	1.22148	1.45349
251275	1066724789	1066976064	23	1.08141	1.45163
1792160394037	31264971026196510	31266763186590547	43	1.26763	1.45051
4915625	25376512	30292137	19	1.07079	1.45008
13	151250	151263	13	1.15682	1.44961
2911801837187	5008072629356125	5010984431193312	41	1.08401	1.44915
567	5000000	5000567	17	1.17362	1.44853
93002175	740353601	833355776	23	1.06856	1.44591
62236321	753494400	815730721	23	1.21957	1.44541
42093683	726383125	768476808	23	1.18422	1.44402
30758507507	82053992493	112812500000	29	1.12654	1.44181
3235661	19000000	22235661	19	1.05157	1.44085
3059	106767578125	106767581184	29	1.1241	1.44085
9526572	12380495	21907067	19	1.05064	1.4404
1	123200	123201	13	1.13692	1.43946
206640625	424928768	631569393	23	1.23636	1.43941

(table continues on next page)

Table 1: Triples with $Q^*(X, Y, Z) > 1.4$ (continued)

X	Y	Z	P(XYZ)	q(X, Y, Z)	Q*(X, Y, Z)
17	610491375	610491392	23	1.20235	1.4386
406847	604661760	605068607	23	1.0519	1.43839
20449	97200	117649	13	1.13245	1.43716
30430647	524733440	555164087	23	1.04743	1.43635
361	19140264	19140625	19	1.26504	1.43632
54925	541458432	541513357	23	1.04613	1.43576
119	3455881	3456000	17	1.14551	1.43569
49	576	625	7	1.20397	1.43546
1244485	2097152	3341637	17	1.34158	1.43451
8704	513288171	513296875	23	1.04335	1.43449
4782969	13152256	17935225	19	1.26013	1.43434
100000000	395598411	495598411	23	1.22156	1.43365
26624	16468459	16495083	19	1.14784	1.43178
77	3070548	3070625	17	1.13651	1.43152
9157915809	309811328125	318969243934	31	1.15726	1.4313
10895347810546875	110733307750904933	121628655561451808	47	1.15167	1.43069
972	2923235	2924207	17	1.1328	1.42979
805376	2088025	2893401	17	1.13199	1.42941
3584	14641	18225	11	1.26669	1.42841
16093	14348907	14365000	19	1.02441	1.42752
607985369874432	2925895385116193	3533880754990625	43	1.16969	1.42693
47183616028125	651166029845027	698349645873152	41	1.2288	1.42651
95985463	269440587	365426050	23	1.02567	1.42631
113379904	239483061	352862965	23	1.02385	1.42546
11921770720625	2888880555709071	2900802326429696	43	1.33358	1.42473
2299	13107200	13109499	19	1.1589	1.42469
98936203248000000	6959907875844540383	7058844079092540383	53	1.00968	1.42451
106832896	225317729	332150625	23	1.16617	1.42399
128000	331533891	331661891	23	1.13558	1.42395
2861001	9938999	12800000	19	1.01724	1.42394
115600000	215222063	330822063	23	1.0205	1.42389
16384	2457945	2474329	17	1.12009	1.42381
6336530479	196089115625	202425646104	31	1.23183	1.42373
190969	321911031	322102000	23	1.13389	1.42324
28647703	290119401	318767104	23	1.11164	1.42298
49	16335	16384	11	1.25294	1.42159
1	11859210	11859211	19	1.22892	1.42156
7168	78125	85293	13	1.43501	1.42082
51	11413325	11413376	19	1.01011	1.42036
4225	165524477184	165524481409	31	1.14015	1.42034

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Table 1: Triples with $Q^*(X,Y,Z) > 1.4$ (continued)

X	Y	Z	P(XYZ)	q(X, Y, Z)	Q*(X, Y, Z)
23	163576413252	163576413275	31	1.10065	1.42014
6030786363476	12098754652149	18129541015625	37	1.03014	1.42014
6178576244914944	4146446969968853449	4152625546213768393	53	1.19892	1.41986
1701	11075584	11077285	19	1.18486	1.41942
321204384157	1702133534151680	1702454738535837	43	1.2991	1.41871
110789	10562500	10673289	19	1.00595	1.41826
2279921	255234375	257514296	23	1.00746	1.41774
4159375	6311981	10471356	19	1.00476	1.41765
25390625	29364256768	29389647393	29	1.22001	1.41762
19	10440125	10440144	19	1.1428	1.41756
1048576	9261945	10310521	19	1.21836	1.41717
17320303	229582512	246902815	23	1.00528	1.4167
33	2000000	2000033	17	1.37156	1.4161
464517838916015625	2094643907511056887	2559161746427072512	53	1.17121	1.41557
348381	234881024	235229405	23	1.15716	1.4155
2080318710	26022962220136321	26022964300455031	47	1.0874	1.41511
1570799616	23447265625	25018065241	29	1.37925	1.41464
9595703125	107307307008	116903010133	31	1.1248	1.41442
6365529	24219934375	24226299904	29	1.3774	1.41404
19096587	24120639488	24139736075	29	1.05829	1.41397
3304765080687213	19280458326171875	22585223406859088	47	1.21006	1.41364
24	1830101	1830125	17	1.09714	1.41285
684648008	102992171875	103676819883	31	1.09372	1.41236
11607334375	89295036953	100902371328	31	1.1183	1.41189
1223089282534721	17575944949625535	18799034232160256	47	1.06276	1.41174
714025	1057536	1771561	17	1.2849	1.41166
94633984	20870760141	20965394125	29	1.05205	1.41134
7291177821	16010150128149250	16010157419327071	47	1.09537	1.41007
1682931712	8409189453125	8410872384837	37	1.11501	1.40955
142706697562835	604146480770349	746853178333184	43	1.01695	1.40923
5437644738767059	7136911973154816	12574556711921875	47	1.09635	1.40754
65045946248063	68889524455062	133935470703125	41	1.20607	1.40651
169996288	15297508265	15467504553	29	1.03858	1.4056
3627679861	65091796875	68719476736	31	1.05616	1.40522
787968	68337783437	68338571405	31	1.1011	1.40512
323486328125	798469078289340003	798469401775668128	53	1.2006	1.40505
28322467313	38654705664	66977172977	31	1.37697	1.40477
30850363685773	456381336042000	487231699727773	43	1.10655	1.40422
4459	1445000	1449459	17	1.0794	1.40422
6184976	5686859049399	5686865234375	37	1.11806	1.40405

(table continues on next page)

Table 1: Triples with $Q^*(X,Y,Z) > 1.4$ (continued)

63026250	85009639	148035889	23	1.15568	1.40387
14641	46875	61516	13	1.06955	1.40374
292500	1127357	1419857	17	1.07783	1.40345
19	145546856	145546875	23	1.34334	1.40344
578000	823543	1401543	17	1.07684	1.40296
729	1398760	1399489	17	1.07673	1.40291
10378389	5228007812500	5228018190889	37	1.11769	1.40286
121	59904	60025	13	1.06717	1.40244
4292352	134886415	139178767	23	1.11448	1.4023
389065088975805	7069506863901443	7458571952877248	47	1.06823	1.40201
517844395629	4344186854371	4862031250000	37	1.11492	1.40183
4231249	131747672	135978921	23	1.06327	1.4017
10000	49049	59049	13	1.06558	1.40157
109844993185235	270577273931757	380422267116992	43	1.24562	1.4013
175206927193540571	349776265604236064	524983192797776635	53	1.00834	1.40118
956630375	6888998370474649	6888999327105024	47	1.03309	1.40116
20776251532943	327683633221350	348459884754293	43	1.06826	1.40025

It is of interest to highlight for fixed primes $P \geq 3$ the champion triples X, Y, Z with $P(XYZ) = P$, being the ones with the largest Z. Table 2 below lists the candidate champions that I found.

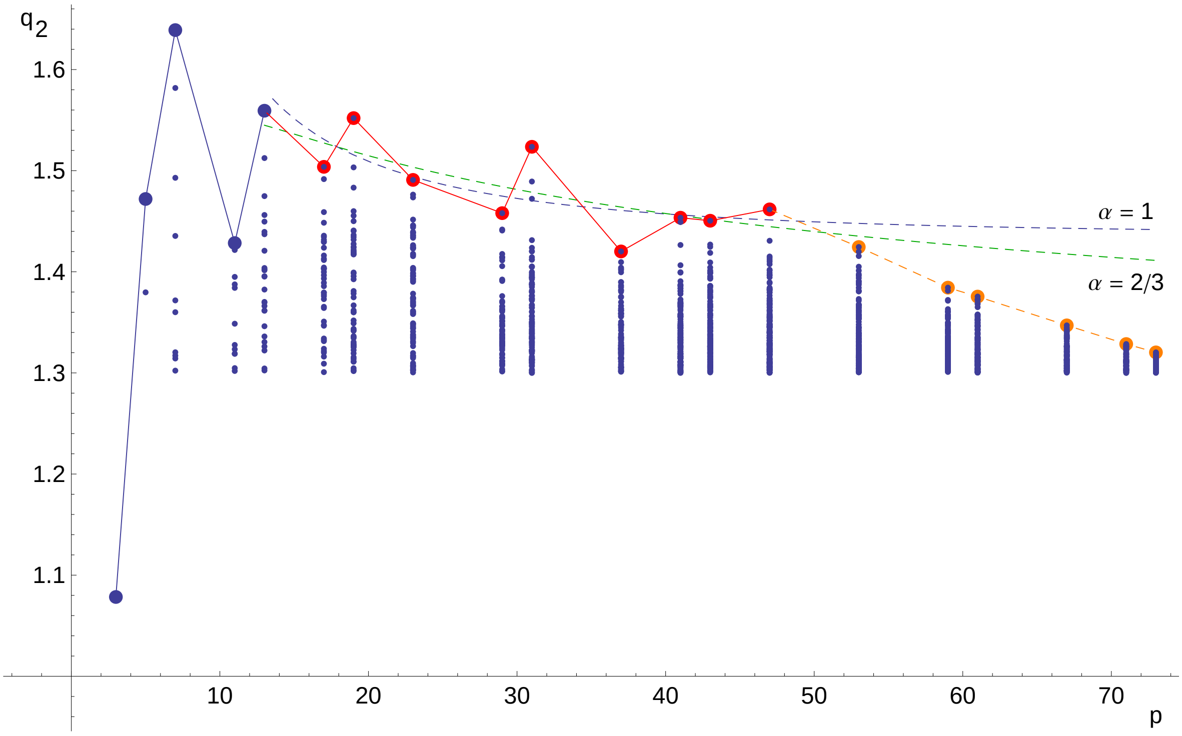
Table 2: Champions for $P \geq 3$

P	X	Y	Z	q(X,Y,Z)	$Q^*(X,Y,Z)$
3	1	8	9	1.22629	1.07480
5	3	125	128	1.42657	1.47200
7	1	4374	4375	1.56789	1.63904
11	3584	14641	18225	1.26669	1.42841
13	91	1771470	1771561	1.39548	1.55930
17	78125	27217619	27295744	1.42158	1.50379
19	4026275	1371299293	1375325568	1.30796	1.55200
23	1446484375	4867359029	6313843404	1.17390	1.49089
29	131769	289478257991	289478389760	1.16826	1.45801
31	3466384181	162615350840320	162618817224501	1.25738	1.52362
37	6030786363476	12098754652149	18129541015625	1.03014	1.42014
41	1169745467950813	6231375784189187	7401121252140000	1.22148	1.45349
43	1792160394037	31264971026196510	31266763186590547	1.26763	1.45051
47	176202799695875	3178472661789594624	3178648864589290499	1.13327	1.46173
53	98936203248000000	6959907875844540383	7058844079092540383	1.00968	1.42451
59	541414815631625728	4588942930987474367	5130357746619100095	1.01356	1.38433
61	1836808757	6804818432899407855	6804818434736216612	1.16851	1.37549
67	8937568294356575	8917128073006643617	8926065641301000192	1.01757	1.34702
71	1734970328918411	8773354729938878464	8775089700267796875	1.09700	1.32856
73	345361623363669913	8819867258522906727	9165228881886576640	1.08854	1.32031

From [dW, Theorem 5.4] it follows that for $P \leq 13$ the values mentioned in Table 2 are indeed the real champions. For $P > 13$ the existence of better examples is not excluded. I conjecture, based on Table 2, that for $P \leq 43$ none exists. When $P = 47$ the champion value of Z might very well surpass the limit of 10^{18} (or even 2^{63}) that is inherent in the "ABC@home" database, and for $P \geq 53$ it most probably does so. Note that at least the candidate champion values for $Q^*(X, Y, Z)$ given in Table 2 are lower bounds for the real champion values.

Figure 1 below shows a graph of $Q^*(X, Y, Z)$ against P, highlighting the found (candidate) champions. Let α be a conjectured value for $\limsup_{Z\to\infty} \frac{\log\log Z}{\log P(XYZ)}$ (so the strong xyz-conjecture of [LS] is that $\alpha = \frac{2}{3}$; also $\alpha = 1$ might be of interest as the abc-conjecture implies $\alpha \leq 1$, see [LS]). This suggests an approximate linear relation between P^α and $\log Z$ for the champions. For what it's worth, for $13 \leq P \leq 47$ and $\alpha = \frac{2}{3}$ as well as $\alpha = 1$ I computed regression lines and extrapolated them up to $P \leq 73$. These lines are shown in Figure 1 as well. I gladly leave it to the reader to decide what to believe for the 'true' value of α.

Figure 1: The found triples with $Q^*(X, Y, Z) > 1.3$, highlighting the champions, and regression lines for the champions



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