Integrals and Valuations

Thierry Coquand

Computing Science Department, Göteborg University

Bas Spitters

Mathematics Department, Eindhoven University of Technology

May 27, 2008

Abstract. We construct a homeomorphism between the compact regular locale of integrals on a Riesz space and the locale of (valuations) on its spectrum. In fact, we construct two geometric theories and show that they are biinterpretable. The constructions are elementary and mostly consist of explicit manipulations on a distributive lattice associated to a given Riesz space.

1. Introduction

The goal of this paper is to give a constructive formulation of the Riesz representation theorem. The Riesz representation theorem states that there is an isomorphism between *integrals* and *regular measures* on compact spaces. An integral on X is a positive linear functional $I: C(X) \to \mathbb{R}$ (and we shall consider only maps such that $I(1) = 1$). A regular measure, or *valuation*, on X is a map $\mu: O(X) \to [0, 1]$ which is monotone, if $U \subseteq V$ then $\mu(U) \leqslant \mu(V)$, and such that $\mu(\emptyset) = 0$ and $\mu(U \cap V) + \mu(U \cup V) = \mu(U) + \mu(V)$ and $\mu(X) = 1$. The regularity condition states that $\mu(U)$ is the sup of $\mu(V)$ for V well-inside U (i.e. such that U contains the closure of V). An equivalent way to express this condition is to state the *continuity* property: if V_i is a directed family then $\mu(\bigcup V_i) = \sup \mu(V_i)$. Such continuous valuations extend uniquely to Borel measures; see [AMJK04] for an overview.

From a constructive point of view there is a crucial difference between the two notions. The integral $I(f)$ of a function $f \in C(X)$ is a *Dedekind real*. Intuitively, this means that one can compute arbitrary rational approximations. This may not be the case for the valuation $\mu(U)$ of an open U: in general we don't have the property that for $r < s$,

$$\mu(U) < s \vee r < \mu(U).$$

Constructively the valuation $\mu(U)$ is only a *lower real*, and can be thought of as a predicate $r < \mu(U)$ on the rationals. This predicate is downward closed: if $r < \mu(U)$ and $s \leq r$ then we have $s < \mu(U)$, but in general, given $\epsilon > 0$ we are not given a way to compute a rational ϵ approximation of $\mu(U)$. Given an integral I we can define a corresponding valuation $\mu_I(U)$ by taking the sup of $I(f)$ over all $0 \leqslant f \leqslant 1$ the support of which included in U. It is remarkable that for *any* valuation μ one can conversely find an (unique) integral I such that $\mu = \mu_I$. So despite the fact that one may not be able to compute $\mu(U)$, it is still possible to compute $\int f d\mu$ as a *Dedekind real* as the supremum of

$$\sum s_i \mu(s_i < f < s_{i+1})$$

over all partitions $s_0 < \cdots < s_n$ of the range $f([a, b])$. A priori this supremum will only be a *lower real*.

As usual in constructive mathematics all structures carry a natural, but implicit, topology and all constructions are continuous. To make this structure explicit we start from a Riesz space R and associate three formal spaces to it that are all compact regular: the maximal spectrum $\mathrm{Max}(R) = X$ (intuitively, R is then a dense subset of $C(X)$), the space of integrals $\mathrm{INT}(R)$ and the space of valuations $\mathrm{VAL}(R)$. All three spaces are defined as *propositional geometrical theories*. A geometric formula is one of the form $\psi \Rightarrow \varphi$, where the formulas ψ and φ are positive, *i.e.* they are built up from atomic formulas using only (finite) conjunction, (infinite) disjunction. A geometric theory is a theory all of which axioms are geometric. The main point of this paper is to define two interpretability maps, showing how to interpret the theory $\mathrm{VAL}(R)$ in the theory $\mathrm{INT}(R)$ (intuitively how to define the measure from an integral) and how to interpret the theory $\mathrm{INT}(R)$ in the theory $\mathrm{VAL}(R)$ (intuitively how to define the measure from the integral). The Riesz representation theorem can then be stated as the fact that these two maps define an isomorphism between the corresponding formal spaces $\mathrm{VAL}(R)$ and $\mathrm{INT}(R)$. This isomorphism is a homeomorphism when the topology defined by the geometric theory $\mathrm{INT}(R)$ is the weak topology. Hence we arrive at a concrete constructive statement of the Riesz representation theorem which is valid in any topos.

The present article is part of our program to apply the logical approach to abstract algebra [CL06] to (functional) analysis [Coq05, CS05, Spi05, Coq06]. It may be seen as a contribution to Hilbert's program of logically translating the use of infinitary methods to finitary, or constructive, ones. It is also continuation of a tradition in topos theory, e.g. [BM06], but in a more explicit manner.[1] It turns out that our program sometimes gives shorter proofs of more general results than a direct constructive treatment in the sense of Bishop. Moreover, the space of valuations does not naturally carry a metric structure and hence the topological structure, explicit in our presentation, is hidden in Bishop's treatment of the Riesz representation theorem. We emphasize, however, that all our results *are* acceptable by Bishop's standard.

1.1. Formal measure and integration theory. As outlined in [Coq04, Spi05, CP02] a formal theory of measure and integration may be developed along the following lines.

In a usual set-theoretic foundation of measure theory on considers certain functions which are defined to be 'measurable'. Then relative to a measure one identifies all the functions which are equal almost everywhere and obtains a vector lattice L_0 of measurable 'functions'. Instead, one may consider such a vector lattice from the beginning, abstracting from the set-theoretic foundations. The benefits of this approach have been emphasized by Kolmogorov, Caratheodory and von Neumann [Rot01]. In the present article we focus on the theory of integrals defined on formal functions and valuations defined on formal *opens*. For a formal treatment of Borel sets we refer to [CP02, Coq04, Sim07].

The abstract space of functions is captured by a Riesz space (a vector lattice) which we require to have a strong unit[2] An integral is a *continuous* linear functional on the Riesz space. On the other hand, a measure is typically only *lower semi-continuous*. This suggests that an integral will be a map to the *Dedekind* reals, but that a valuation will map to the *lower* reals. The Riesz representation theorem will be presented in the form of a homeomorphism between the formal space of integrals on a Riesz space and valuations on the opens of its spectrum. By the Stone-Yosida theorem any Riesz space R with strong unit can be embedded in the space of continuous functions over its spectrum $\mathrm{Max}(R)$. This can be proved constructively [CS05]. The integral extends to this space of continuous functions. In this sense our approach is close to the Daniell integral.

1.2. Overview. Section 3 contains the statement of the Riesz representation theorem, the main result of the article. The statement is geometric with joins restricted to countable sets. This allows us to use logical methods to conclude classically that there has to be a constructive proof. We construct such a proof in Section 5. The proof uses a concrete theory of non-increasing functions, which we call Δ-functions in Section 4.

2. Preliminaries

2.1. Lower and upper reals.

The lowerreals, denoted $\mathbb{R}^{\mathrm{low}}$, are defined as inhabited, down-closed, open subsets of the rationals. Upperreals are defined similarly. Lower(upper) reals are closed under addition and closed under multiplication by a positive rational. However, the lower and upper reals are not closed under subtraction, but one *can* subtract a lower real from an upper real and obtain an upper real. The non-strict inequality $\leqslant$ is given by inclusion of subsets. In the absence of the powerset operator, the lower reals are better considered as a formal space rather than a set, but we will not emphasize this point.

2.2. Logic and topology.

In set theory, *i.e.* in the topos Set, one uses topological spaces to deal with continuity. However, statements including points of topological spaces are often difficult to generalize to arbitrary toposes. Fortunately, it is often possible to resort to the lattice structure of the open sets of a topological space. These complete distributive lattices are thus called 'pointfree' spaces, or locales (see [Joh82]). In the topos Set one can often reconstruct the points from this lattice; to be precise, there is an adjunction between the category of topological spaces and the category of locales, which restricts to an equivalence of categories between compact Hausdorff spaces and compact completely regular locales. In general, this equivalence is not present in a topos. When generalizing theorems from the topos Set to an arbitrary topos focusing on locales is often the better choice. One reason for this is that a locale may be defined by geometric theory. In logical terms the locale is its syntactic category, often called the Lindenbaum algebra — that is, the poset of provable equivalence classes, ordered by provable

1. We avoid the axiom of (countable) choice, and, moreover, we refrain from using the power set axiom. One may wonder how we treat the set of all real numbers in such a framework. In fact, we do not use this set at all. We only consider the *formal space* of real numbers.

2. An even weaker requirement would have been to demand that we are given an lattice ordered Abelian group. Such a group can be extended to a Riesz space over the rationals [Coq05].

entailment. The correspondence between the locale and the theory is the usual completeness and consistency link between theories and models. The models of the theory correspond to completely prime filters, *i.e.* points of the locale presented by the lattice. In this way, a point x in a topological space defines a model of the corresponding theory: a basic proposition I is true in the model x iff $x \in I$. This view leads us to consider theories as primary objects of study; their models, the points, will be derived concepts. Hence topology is propositional geometric logic; see e.g. [Joh02, Vic07].

2.3. Normal lattices and entailment relations.

In practice, the infinitary disjunctions of geometric logic can often be restricted to concrete countable sets such as the rationals. Even more explicitly, a compact regular locale can be be conveniently represented by a normal distributive lattice [Coq03]. The regular ideals of a normal distributive lattice define a frame which is compact regular. Conversely, every compact regular locale can be presented in this way.

In turn it is occasionally useful to consider a distributive lattice generated by an entailment relation [CC00]. An *entailment relation* on a set S is a reflexive, monotone and transitive relation $\vdash$ between finite subsets of S:

Reflexive. $X \vdash Y$, if $X \cap Y$ is inhabited;

Monotone. If $X \vdash Y$, then $X', X \vdash Y, Y'$;

Transitive. If $X \vdash s, Z$ and $X, s \vdash Z$, then $X \vdash Z$.

Such an entailment relation generates a distributive lattice in which $\bigwedge X \leqslant \bigvee Y$ iff $X \vdash Y$.

2.4. Spectrum of a Riesz space.

Definition 1. *A* Riesz space *is a vector space with a compatible lattice structure. An element* 1 *is a* strong unit *if for all x there exists n such that $-n1 \leqslant x \leqslant n1$.*

In a Riesz space one defines $f^+ := f \vee 0$, $f^- := 0 \vee -f$ and $|f| := f^+ + f^-$ and derives that $f = f^+ - f^-$.

The spectrum of a Riesz space R is the space of all its representations. It may be presented as a compact regular locale, or as the normal distributive lattice freely generated by the collection of tokens $D(a)$, one for each a in R, subject to the following relations:

1. $D(a) = \bot$, if $a \leq 0$;
2. $D(1) = \top$;
3. $D(a) \wedge D(-a) = \bot$;
4. $D(a+b) \leqslant D(a) \vee D(b)$;
5. $D(a \vee b) = D(a) \vee D(b)$.

This lattice represents the locale free generated by the generators and relations above and the continuity axiom $D(a) = \bigvee_s D(a-s)$.

The Stone-Yosida representation theorem states that there is a dense embedding of R into the locale of real valued continuous functions on its spectrum. A real valued *continuous function* on a locale is given by two families L_q, U_q of opens indexed by the rationals. These families satisfy the following relations.

1. $\bigvee U_r = \bigvee L_s = \top$;
2. $U_r = \bigvee_{r' > r} U_{r'}$, $L_r = \bigvee_{s' < s} L_{s'}$;
3. $U_r \vee L_s = \top$ if $r < s$;
4. $U_r \wedge L_s = \bot$ if $s \leqslant r$.

Intuitively, U_r stands for $f^{-1}(r, \infty)$ and L_s stands for $f^{-1}(-\infty, s)$. An *integral* on a Riesz space is a positive linear functional. By density, an integral extends uniquely to a positive linear functional on the space of all continuous real-valued functions on the spectrum.

3. STATEMENT OF THE RIESZ-REPRESENTATION THEOREM

The goal of this section is to state, in Subsection 3.3, the Riesz representation theorem as the existence of a continuous bijection between the formal compact regular spaces of integrals and valuations. Theorem 24 contains the proof of the representation theorem.

3.1. The space of integrals.

Let R be a Riesz space with strong unit 1. We present a theory INT of integrals on R, much like the description of the Stone's maximal spectrum $\mathrm{Max}(R)$ above. Subbasic opens $[p < I(f)]$ are indexed by $p > 0$ and f in R. The set of its points is $\{I \mid p < I(f)\}$. Since $p < I(f)$ iff $0 < I(f - p)$, it is sufficient to treat basic opens of the form $0 < I(f)$, written $P(f)$, where P is a dummy symbol. The points in this open are integrals I such that $0 < I(f)$. The distributive lattice INT has generators $P(f)$ and relations:

I.1. $\top = P(1)$;

I.2. $P(f) \wedge P(-f) = \bot$;

I.3. $P(f + g) \leqslant P(f) \vee P(g)$;

I.4. $P(f) = \bot$ if $f \leqslant 0$.

This lattice may also be seen as a theory with propositions $P(f)$ and the axioms defined above. This theory was studied in [Coq05] as the theory TOT of total orderings on an ordered vector space.

The rule $P(f) \wedge P(g) \leqslant P(f + g)$ can be derived in INT as follows: Since $f = f + g - g$, $P(f) \leqslant P(f + g) \vee P(-g)$, taking the meet with $P(g)$ on both sides and applying distributivity we obtain the rule.

Definition 2. *A distributive lattice is* normal *if for all b_1, b_2 such that $b_1 \vee b_2 = \top$ there are c_1, c_2 such that $c_1 \wedge c_2 = \bot$ and $c_1 \vee b_1 = \top$ and $c_2 \vee b_2 = \top$. In a distributive lattice we define $a \ll b$ if there exists c such that $c \vee b = \top$ and $a \wedge c = \bot$. A distributive lattice is called* strongly normal *when for all a, b there exist x, y such that $a \leqslant b \vee x$ and $b \leqslant a \vee y$ and $x \wedge y = \bot$.*

To check that a lattice is strongly normal it is enough to check it for a, b among the generators of the lattice.

Lemma 3. *Every strongly normal lattice is normal.*

Proof. Let $b_1 \vee b_2 = \top$. Choose x, y such that $b_1 \leqslant b_2 \vee x$ and $b_2 \leqslant b_1 \vee y$ and $x \wedge y = \bot$. Then $\top \leqslant b_1 \vee b_2 \leqslant (b_2 \vee x) \vee b_2 = b_2 \vee x$. Similarly, $\top = b_1 \vee y$. □

Lemma 4. *The lattice INT is strongly normal.*

Proof. Choose f, g in R. Then
$P(f) \leqslant P(f - g) \vee P(g)$;
$P(g) \leqslant P(g - f) \vee P(f)$;
$P(f - g) \vee P(g - f) = \bot$. □

We derive a few simple facts.

It follows from **I.3** that $P(f) \leqslant P(\frac{1}{n} f)$ and $P(f) \leqslant P(m f)$ (since $P(2f - f) \leqslant P(2f) \vee P(-f)$ and $P(-f) = \bot$, etc.). Thus $P(f) \leqslant P(q f)$ for all $q \in \mathbb{Q}^+$. We define $[p < I(f)]$ as $P(f - p)$ and define $[I(f) < q]$ as $P(q - f)$. Then

- $P(f + \varepsilon) = \top$ if $f \geqslant 0$ and $\varepsilon > 0$.
  Proof: $P(f + \varepsilon - f) \leqslant P(f + \varepsilon) \vee P(-f)$ and $P(-f) = \bot$.
- $\top = [s < I(f)] \vee [I(f) < t]$, whenever $s < t$.
  Proof: $P(t - s) = \top$.
- $P(f) \leqslant P(g)$ whenever $f \leqslant g$.

Proposition 5. $\bigwedge_X P(f_i) \leqslant \bigvee_Y P(g_j)$ *iff there exists rational r, r_i, $s_j \geqslant 0$ such that $r + \sum_X r_i f_i \leqslant \sum_Y s_j g_j$ and $r + \sum r_i > 0$.*

Proof. We use the technique of entailment relations [CC00].

The relation on the right hand side is an entailment relation.

Reflexive. If $f \in X \cap Y$, then $f \leqslant f$;

Monotone. If $r + \sum_X r_i f_i \leqslant \sum_Y s_j g_j$, then $r + \sum_X r_i f_i + \sum_{X'} 0 f_i \leqslant \sum_Y s_j g_j + \sum_{Y'} 0 g_j$;

Transitive. If $r + \sum_X r_i f_i \leqslant \sum_Y s_j g_j + s g$ and $r + \sum_X r_i f_i + s' g \leqslant \sum_Y s_j g_j$, then, assuming $s' \neq 0$, $\frac{s}{s'}(r + \sum_X r_i f_i + s' g) \leqslant \frac{s}{s'} \sum_Y s_j g_j$ and so $(1 + \frac{s}{s'})(r + \sum_X r_i f_i) \leqslant (1 + \frac{s}{s'}) \sum_Y s_j g_j$.

Conversely, suppose that $r + \sum r_i f_i \leqslant \sum s_j g_j$. Then

$$\bigwedge P(f_i) \leqslant \bigwedge_{r_i \neq 0} P(r_i f_i) \leqslant P(r + \sum r_i f_i) \leqslant P(\sum s_j g_j) \leqslant \bigvee P(g_j). \qquad \square$$

Lemma 6. *$u \ll P(g)$ iff for some $s > 0$, $u \leqslant P(g - s)$.*

Proof. Suppose that $u \ll P(f)$. Then there exists v, $u \wedge v = \bot$ and $P(f) \vee v = \top$. We write $v = \bigwedge \bigvee P(f_{ij})$. Then for each i, $P(f) \vee \bigvee_j P(f_{ij}) = \top$. By the previous proposition there exists $s_i > 0$ such that $P(f - s_i) \vee \bigvee_j P(f_{ij}) = \top$. Taking the maximum over all s_is finishes the proof. $\square$

In the following two paragraphs we connect for the benefit of the reader the present development with the theory of locales. However, these results are not strictly needed in the rest of the article.

The locale of integrals can be defined to be the retract of the normal lattice INT and hence is compact regular by construction. The ordering relation $a \leqslant b$ on the locale is defined by for all $a' \ll a$, $a' \leqslant b$ in the lattice. By Lemma 6, the locale of integrals is obtained as the locale freely generated by adding the following continuity relation:

$$P(f) \leqslant \bigvee_{s \in \mathbb{Q}^+} P(f - s)$$

to the theory INT. Consequently, a model I of the theory corresponds to an integral defined by $I(f) := \sup \{s \mid I \vDash [s < I(f)]\}$. This integral takes its values in the Dedekind real numbers.

Usually, one proves that the space of integrals is compact by an appeal to the Alaoglu theorem which depends on the Tychonoff theorem. Here we have shown that it is compact by construction. A similar construction can be carried out for Tychonoff's theorem, not only for compact regular locales, but also for general compact locales [Coq92][Joh81].

Instead of starting with a positive linear functional, it will later be convenient to work with its restriction to the positive elements. This theory is called INTPOS. The corresponding lattice is generated by the symbols

- **IP1.** $[0 < I(1)] = \top$; $[1 < I(1)] = \bot$;
- **IP2.** $[s < I(f)] \vee [I(f) < t] = \top$ whenever $s < t$;
- **IP3.** $[t < I(f)] \vee [I(f) < t] = \bot$;
- **IP4.** $[s + t < I(f + g)] \leqslant [s < I(f)] \vee [t < I(g)]$;
- **IP5.** $[s < I(f)] \leqslant [s < I(g)]$ whenever $f \leqslant g$;
- **IP6.** $[I(g) < t] \leqslant [I(f) < t]$ whenever $f \leqslant g$.

Where $f, g \geqslant 0$ and s, t are non-negative rationals. The axioms **IP2/3** state that the integral produces a Dedekind real number. Axiom **IP4** is linearity of the integral. The axioms **IP5/6** state that the integral is positive.

Proposition 7. *The theories INT and INTPOS are biinterpretable.*

Proof. We have already shown how to interpret INTPOS in INT. We will now consider the converse.

We interpret $P(f)$ as $\bigvee_s [I(f^-) < s] \wedge [s < I(f^+)]$. Only the axiom **I.3** needs attention. This is derived from the equation $(f + g)^+ + f^- + g^- = f^+ + g^+ + (f - g)^-$: If $I((f - g)^-) < I((f + g)^+)$, then either $I(f^-) < I(f^+)$ or $I(g^-) < I(g^+)$. Consequently, if $P(f + g)$, then $P(f)$ or $P(g)$. $\square$

The argument remains valid when we add the continuity rule to both sides.

3.2. The space of valuations.

We define the space of valuations on a distributive lattice L as the theory VAL. The generators are $[p < \mu(x)]$ where x in L and p in $\mathbb{R}$.

- **V.1.** $[q < \mu(\top)] = \top$ if $q < 1$; $[q < \mu(\top)] = \bot$ if $q \geqslant 1$;
- **V.2.** $[q < \mu(\bot)] = \bot$ if $0 \leqslant q$; $[q < \mu(\top)] = \top$ for $q < 0$;
- **V.3.** $[p < \mu(x)] \wedge [q < \mu(y)] \leqslant ([a < \mu(x \vee y)]) \vee ([b < \mu(x \wedge y)])$ if $a + b = p + q$;
  $([a < \mu(x \vee y)]) \wedge ([b < \mu(x \wedge y)]) \leqslant [p < \mu(x)] \vee [q < \mu(y)]$ if $a + b = p + q$;
- **V.4.** $[p < \mu(x)] \leqslant [q < \mu(y)]$ if $x \leqslant y$ and $q \leqslant p$.

Lemma 8. *If* $b \vee m = 1$ *and* $s + t < 1$, *then* $[s < \mu(b)] \vee [t < \mu(m)] = \top$.

Proof. $\mu(b \vee m) = \mu(\top) > s + t$. The result now follows from modularity (**V.3**). □

The following construction will be used in the next proof, but also used later. Let L be a distributive lattice. For $(x_i)_{i \in I}$ in L define $x_J := \wedge x_j$ where J a finite subset of I. Following Tarski [Tar38] we define the ordered monoid $M(L)$ of formal sums $\sum n_i x_i$, x_i in L. The order is $\sum x_i \leqslant \sum y_j$ iff $x_I \leqslant \bigvee \{y_J \mid |J| = k\}$ whenever $|I| = k$. The monoid $M(L)$ satisfies the cancellation property. Moreover for $k > 0$, $k.x \leqslant 0$ iff $x = 0$. We add positive rational coefficients — that is, define a relation $\sum r_i x_i \leqslant \sum s_j y_j$ — by putting all the terms on one numerator. When L is normal, it can be shown that if $1 \leqslant rx + m$, m in $M(L)$, then there exists $x' \ll x$ such that $1 \leqslant rx' + m$. If r in $\mathbb{Q}^+$ and $x \leqslant y$, then $rx \leqslant ry$ and $x + z \leqslant y + z$. When L is a lattice of sets, this coincides with the usual ordering of simple functions. We return to this interpretation in Section 5.1.

Proposition 9. *In the lattice VAL,* $\bigwedge\ [p_i < \mu(x_i)] \leqslant \bigvee\ [q_j < \mu(y_j)]$ *iff there exists* $r, r_i, s_j, s \geqslant 0$ *and* $p < 1$ *such that* $r + \sum r_i = 1$, $s + \sum\ s_j q_j \leqslant rp + \sum\ r_i p_i$ *and* $r + \sum\ r_i x_i \leqslant \sum\ s_j y_j + s$ *in* $M(L)$.

Proof. We use the technique of entailment relations [CC00].

Suppose that there exists $r, r_i, s_j, s \geqslant 0$ and $p < 1$ such that $r + \sum r_i = 1$, $s + \sum s_j q_j \leqslant rp + \sum\ r_i p_i$ and $r + \sum\ r_i x_i \leqslant \sum\ s_j y_j + s$. Then, using the suggestive notation $[\sum\ p_i < \mu(\sum\ x_i)]$, to be made precise in Section 5.1, we have

$$\begin{aligned}\bigwedge\ [p_i < \mu(x_i)] = \bigwedge\ [p_i < \mu(x_i)] \wedge [p < \mu(\top)] &\leqslant [rp + \sum\ r_i p_i < \mu(r + \sum\ r_i x_i)] \\ &\leqslant [s + \sum\ s_j q_j < \mu(s + \sum\ s_j y_j)] \\ &\leqslant \bigvee\ [s_j q_j < \mu(s_j y_j)] \vee [s < \mu(s)] \\ &\leqslant \bigvee\ [\,q_j < \mu(y_j)].\end{aligned}$$

The first inequality follows by a repeated application of **V.3**. The second inequality from the definition of the order on simple functions and **V.4**.

The relation on the right hand side is reflexive, monotone and transitive. It also validates all the axioms **V.1-4**. It is thus an entailment relation extending the one of the lattice.

We conclude that the relation coincides with the entailment relation in the lattice. □

Proposition 10. *If* L *is normal, then is the lattice defined by the theory VAL.*

Proof. We start with some computations.

Suppose that $\top \leqslant [q_0 < \mu(y_0)] \vee [q_1 < \mu(y_1)]$. Then there exists $s_j, s \geqslant 0$ and $p < 1$ such that $s + \sum s_j q_j \leqslant p$ and $1 \leqslant \sum\ s_j y_j + s$. The second relation implies that $y_0 \vee y_1 = \top$. Moreover, $\sum\ s_j q_j \leqslant p - s < 1 - s \leqslant \sum\ s_j y_j$. This implies that $q_0 + q_1 < 1$, since $q_0 + q_1 \geqslant 1$ is impossible.

Suppose that $[p_0 < \mu(x_0)] \wedge [p_1 < \mu(x_1)] \leqslant \bot$ and $p_0, p_1 < 1$. Then there exists $r, r_i, s \geqslant 0$ and $p < 1$ such that $r + \sum\ r_i = 1$, $s \leqslant rp + \sum\ r_i p_i$ and $r + \sum\ r_i x_i \leqslant s$. So, $\sum\ r_i x_i \leqslant s - r < s - rp \leqslant \sum\ r_i p_i$. By the definition of the order on simple functions this means that $x_0 \wedge x_1 = \bot$ or $r_0 + r_1 \leqslant \sum\ r_i p_i$, in the latter case: $p_0 \geqslant 1$ or $p_1 \geqslant 1$, contradicting the assumption. So we may assume that $x_0 \wedge x_1 = \bot$ and hence $\vee r_i \leqslant \sum\ r_i p_i$, so $p_0 + p_1 \geqslant 1$. Conversely, this implies that $[p_0 < \mu(x_0)] \wedge [p_1 < \mu(x_1)] \leqslant \bot$.

To prove normality of the theory VAL suppose that $\top \leqslant [q_0 < \mu(y_0)] \vee [q_1 < \mu(y_1)]$. As we showed above, $y_0 \vee y_1 = \top$ and $q_0 + q_1 < 1$. By the normality of L, there are c_0, c_1 such that $c_0 \wedge c_1 = \bot$ and $c_0 \vee y_0 = \top$ and $c_1 \vee y_1 = \top$. Hence the opens $[1 - q_0 < \mu(c_0)]$ and $[1 - q_1 < \mu(c_1)]$ witness that the lattice is normal. □

Lemma 11. $u \ll [p < \mu(D(a))]$ *iff for some* $q > p$ *and* $\varepsilon > 0$, $u \leqslant [q < \mu(D(a - \varepsilon)]$.

Proof. Suppose that $u \ll P(f)$. Then there exists v, $u \wedge v = \bot$ and $[p < \mu(D(a))] \vee v = \top$. We write $v = \bigwedge\ \bigvee\ [p_{ij} < \mu(D(a_{ij}))]$. Then for each i, $[p < \mu(D(a))] \vee \bigvee_j\ [p_{ij} < \mu(D(a_{ij}))] = \top$. By the previous propositions there exists $\varepsilon_i > 0$ and q_i such that $[q_i < \mu(D(a - \varepsilon_i))] \vee \bigvee_j\ [p_{ij} < \mu(D(a_{ij}))] = \top$. Taking the minimum over all q_is and the minimum over all ε_is finishes the proof. □

The normal distributive lattice VAL defines a locale also called VAL. The order relation $a \leqslant b$ of this locale is defined by: for all $a' \ll a$, $a' \leqslant b$ in the lattice VAL. The locale is thus the one generated by the axioms **V1-4** together with the axioms

V.5. $[p < \mu(x)] \leqslant \bigvee_{q > p}\ [q < \mu(x)]$;

Reg. $[p<\mu(D(a))]\leqslant\bigvee_{\varepsilon>0}[p<\mu(D(a-\varepsilon)]$.

A point of this locale, a model of the theory, is a valuation — that is, a map from the locale generated by the normal distributive lattice L to the lower reals such that $\mu(x)+\mu(y)=\mu(x\vee y)+\mu(x\wedge y)$.

By deriving the relation $\ll$ from the logical description, we have derived the natural topology on the set of valuations; see also [MJ02] where a similar language is used.

3.3. Statement of the theorem.

We are now ready to define the promised maps between integrals and valuations. The spaces are defined concretely as geometric theories, so an interpretation of one theory in the other induces a frame map, and hence a locale map in the opposite direction.

From integrals to valuations.

Given an integral on a Riesz space, we construct a valuation on the opens in its spectrum. We define an interpretation of VAL in INT — that is, a map from integrals to valuations,

$$\mathrm{IV}([p<\mu(D(a))]):=\bigvee_N\{[p<I(Na^+\wedge 1)]\mid N\in\mathbb{N}\}=\bigvee_{f,N}\{[p<I(f)]\mid 0\leqslant f\leqslant Na^+\}.$$

From valuations to integrals.

In order to define the converse interpretation we introduce some notations. Write $(r',s')\ll(r,s)$ for $r<r'<s'<s$. Let f be in R. Define $[p<\Delta_f(r,s)]:=[p<\mu(s<f<r)]$. More precisely,

$$[p<\Delta_f(r,s)]:=\bigvee_{\{p_1,p_2\mid p_1+p_2=p+1\}}[p_1<\mu(s<f)]\wedge[p_2<\mu(f<r)].$$

Let $I=(r,s)$. Write $\Delta_f(I^c)$ for the lower real $\Delta_f(-\infty,r)+\Delta_f(s,\infty)$. More formally,

$$[p<\Delta_f(I^c)]:=\bigvee_{a+b\geqslant p}[a<\Delta_f(-\infty,r)]\vee[b<\Delta_f(s,\infty)].$$

Write $\Delta_f[I]$ for the upper real $1-\Delta_f(I^c)$. More formally, $[\Delta_f[I]<p]:=[1-p<\Delta_f(I^c)]$.

Lemma 12. *If $I\ll J$ and $p<q$, then* $\top=[p<\Delta_f(J)]\vee[\Delta_f[I]<q]$.

Proof. Since $\Delta_f(I^c)+\Delta_f(J)\geqslant 1>p+(1-q)$. □

The interpretation of INTPOS in VAL

$$\begin{aligned}\mathrm{VI}([p<I(f)]) &:= \bigvee(s_i).[\sum s_i\Delta_f(s_i,s_{i+1})>p]\\ \mathrm{VI}([I(f)<q]) &:= \bigvee(s_i).[\sum s_{i+1}\Delta_f[s_i,s_{i+1}]<q]\end{aligned}$$

Again this is a disjunction over a concrete countable set: a finite list of strictly increasing rationals.

Assuming the classical Riesz representation theorem it is easy to show that these are indeed interpretations and that these maps are each others inverses as follows: For any $r>0$ there is an r-approximation by sums $\sum s_i\Delta_f(s_i,s_{i+1})$ and $\sum s_{i+1}\Delta_f[s_i,s_{i+1}]$. This follows from the usual classical proof of Riesz Theorem and the possibility to choose s_i as continuity points for the function

$$s\mapsto\Delta_f(-\infty,s)$$

By completeness of propositional ω-logic [MR77, ST58] and the validity of the propositions in all models, i.e. measures or integrals, of the theory we see that, classically, there should be a proof in the theory that these are indeed interpretations. We will provide such a constructive proof in Theorem 24.

4. Δ-functions

Before we proceed with the proof of the Riesz representation theorem, we develop a small theory of Δ-functions which represent non-increasing interval valued functions. We will require that such a function α satisfies $\alpha(a')=1$ for all a' less than some a and $\alpha(b')=0$ for all b' bigger than some b. Intuitively, a Δ-function Δ represents essential information about α as the lowerreal $\Delta(s,t)=\alpha(s)_l-\alpha(t)_u$. It seems easier to work with the Δ-function directly instead of working with the function α. The following definition may be read as a theory and hence presents a frame of Δ-functions. Our prime application will be to the function $\Delta(r,s)=\mu(r<f<s)$ for a given f in R and a measure μ. Technically, this can be seen as an interpretation of the theory of Δ-functions in the theory VAL which is an extension of the theory of Riesz spaces.

Definition 13. *A* Δ*-function on* (a,b) *is a function* $\Delta\colon \{(r,s)\in\mathbb{Q}^2 \mid r<s\}\to\mathbb{R}^{\mathrm{low}}$ *such that*

1. $\Delta(a,b)=1$ *for some* $a<b$*;*
2. $\Delta(r,s)\leqslant 1$*;*
3. $\Delta(r,s)\geqslant 0$*;*
4. $\Delta(r,s)+\Delta(s,t)\leqslant\Delta(r,t)$;
5. $\Delta(r',s')\leqslant\Delta(r,s)$ *whenever* $r\leqslant r'<s'\leqslant s$*;*
6. $\Delta(r,s')+\Delta(r',s)=\Delta(r,s)+\Delta(r',s')$ *whenever* $r<r'<s'<s$*.*

When the Δ-function jumps in s we do not have equality in 3 above.

As in Lemma 12 we have:

Lemma 14. *If* $I\ll J$ *and* $p<q$*, then* $\Delta(J)>p$ *or* $\Delta[I]<q$*.*

We now prove 'a non-increasing function is continuous in a dense set of points' in a pointfree way. The existence below expresses that at least one of a finite number of options holds. It can thus be expressed in the theory of Δ-functions.

Theorem 15. *Let* $N\in\mathbb{N}$ *and* $I=(r,s)$ *be an open interval. Then there exists an interval* $J\ll I$ *such that* $\Delta[J]<\frac{1}{N}$*.*

Proof. Choose $2N$ disjoint intervals I_i in I and choose $2N$ intervals $J_i\ll I_i$. For each i, $\Delta(I_i)>\frac{1}{2N}$ or $\Delta[J_i]<\frac{1}{N}$. It is impossible that the former case occurs all the time, therefore the latter case occurs at least once. □

The following proposition defines a map from the Δ-functions to the Dedekind reals. A Dedekind real is a pair (L,U) of a lower real and an upper real such that for all $s<r$, $s\in L$ or $r\in U$. The interpretation of this map is the Stieltjes integral $\int t\,d\alpha(t)$, where α is a non-decreasing function connected to Δ.

Proposition 16. *Let* $\Delta(a,b)=1$*. Then* $(\{p\mid p<\sum s_i\Delta(s_i,s_{i+1})\},\{q\mid \sum s_{i+1}\Delta[s_i,s_{i+1}]<q\})$*, where* s_i *ranges over finite partitions of* $[a,b]$*, defines a Dedekind real.*

Proof. Without loss of generality $a=0$ and $b=1$.

We first prove that the upper and lower cut come arbitrary close: There exists (s_i) such that $\sum s_{i+1}\Delta[s_i,s_{i+1}]-\sum s_i\Delta(s_i,s_{i+1})$ is small. To wit, using Theorem 15 choose a partition s_i of $[0,1]$ such that $\sum\Delta[s_i]$ is small, where we write $\Delta[s_i]:=\Delta[s_i,s_i]$. Then

$$\begin{aligned}\sum s_{i+1}\Delta[s_i,s_{i+1}]-\sum s_i\Delta(s_i,s_{i+1}) &\leqslant \sum (s_{i+1}-s_i)\Delta[s_i,s_{i+1}]\\ &\leqslant \varepsilon\sum\Delta[s_i,s_{i+1}]\\ &\leqslant \varepsilon(1+\sum\Delta[s_i])\end{aligned}$$

is small.

We now prove that the lower set is below the upper set. First, we observe that for $r<s<t$,

$$r(\Delta(r,t)-\Delta[s])\leqslant r(\Delta(r,s)+\Delta(s,t))\leqslant r\Delta(r,s)+s\Delta(s,t).$$

So, if s_i refines t_j, then

$$\sum t_j\Delta(t_j,t_{j+1})-\sum_{s_i\notin\{t_j\}} s_i\Delta[s_i]\leqslant\sum_i s_i\Delta(s_i,s_{i+1}).$$

Similarly,

$$\sum_i s_{i+1}\Delta[s_i,s_{i+1}]\leqslant\sum t_{j+1}\Delta(t_j,t_{j+1})+\sum_{s_i\notin\{t_j\}} s_i\Delta[s_i].$$

When an upper real u and a lower real l are both bounded by a rational number q we can define $l\leqslant u$ as $q-u+l\leqslant q$. With this notation $\sum_i s_i\Delta(s_i,s_{i+1})\leqslant\sum_i s_{i+1}\Delta[s_i,s_{i+1}]$. This completes the proof since $\sum_{s_i\notin\{t_j\}} s_i\Delta[s_i]$ may be made arbitrary small. □

The previous proposition contains the essence of Bishop's profile theorem; see [BB85]. It is the crucial step in the proof that VI is a function, see Lemma 23.

5. Proof of the Riesz representation theorem

In this section we prove the Riesz representation theorem.

5.1. Simple functions.

We define formal simple functions on a a distributive lattice L. For $(x_i)_{i\in I}$ in L define $x_J := \wedge x_j$ where J a finite subset of I. Following Tarski [Tar38] we define the ordered monoid $M(L)$ of formal sums $\sum n_i x_i$, x_i in L. The order is $\sum x_i \leqslant \sum y_j$ iff $x_I \leqslant \bigvee \{y_J \mid |J| = k\}$ whenever $|I| = k$. The monoid $M(L)$ satisfies the cancellation property. For $k > 0$, $k.x \leqslant 0$ iff $x = 0$. We add positive rational coefficients — that is, define a relation $\sum r_i x_i \leqslant \sum s_j y_j$ — by putting all the terms on one numerator. When L is normal, it can be shown that if $1 \leqslant rx + m$, m in $M(L)$, then there exists $x' \ll x$ such that $1 \leqslant r x' + m$. If r in $\mathbb{Q}^+$ and $x \leqslant y$, then $r\,x \leqslant r\,y$ and $x + z \leqslant y + z$. When L is a lattice of sets, this coincides with the usual ordering of simple functions.

We write $r_I := \sum_{i\in I} r_i$.

Lemma 17. $\sum r_i x_i \leqslant \sum s_j y_j$ *iff* $x_I \leqslant \bigvee_{J, r_I \leqslant s_J} y_J$.

Lemma 18. *The relation* $\leqslant$ *is transitive on simple functions.*

Proof. Suppose that $\sum r_i a_i \leqslant \sum s_j b_j \leqslant \sum t_k c_k$. Then $a_I \leqslant \bigvee_{J, r_I \leqslant s_J} b_J$ and $b_J \leqslant \bigvee_{K, s_J \leqslant t_K} c_K$. So, $a_I \leqslant \bigvee_{J,K, r_I \leqslant s_J, s_J \leqslant t_K} c_K$. □

We now specialize to the case where L is the distributive lattice corresponding to the theory of the spectrum of a Riesz space. A real valued function f on a locale is given by a collection of opens U_r, L_r as in Subsection 2.4. We denote the open by U_r by $(f > r)$ and L_r by $(f < r)$. In particular, by the Stone-Yosida representation, any element a of the Riesz space may be presented by such a function the families are then given by $D(r - a)$ and $D(a - s)$. Define the relation $\sum r_i a_i \leqslant f$ as: for all I, $a_I \leqslant (r_I < f)$. Define the relation $f < \sum s_j b_j$ as: $1 = \bigvee_J ((f < s_J) \wedge b_J)$. The construction of simple functions is naturally extended to the Boolean algebra generated by the lattice [CC00]. In this Boolean algebra the expression $(f \leqslant t)$ denotes the formal complement of $(t < f)$.

Lemma 19. *Suppose that* $\sum r_i a_i \leqslant \sum s_j b_j$ *and* $\sum s_j b_j \leqslant f$. *Then* $\sum r_i a_i \leqslant f$.

Proof. We have $a_I \leqslant \bigvee_{J, r_I \leqslant s_J} b_J$ and $b_J \leqslant (s_J < f)$. So

$$a_I \leqslant \bigvee_{J, r_I \leqslant s_J} (s_J < f) \leqslant \bigvee_{J, r_I \leqslant s_J} (r_I < f) \leqslant (r_I < f).$$ □

Lemma 20. *Suppose that* $f < \sum r_i a_i$ *and* $\sum r_i a_i \leqslant \sum s_j b_j$. *Then* $f < \sum s_j b_j$.

Proof. We have $a_I \leqslant \bigvee_{J, r_I \leqslant s_J} b_J$ and $b_J \leqslant (s_J < f)$. So

$$a_I \leqslant \bigvee_{J, r_I \leqslant s_J} (s_J < f) \leqslant \bigvee_{J, r_I \leqslant s_J} (r_I < f) \leqslant (r_I < f).$$ □

It is clear that if $\sum r_i a_i \leqslant f \leqslant g$, then $\sum r_i a_i \leqslant g$, and if $f \leqslant g < \sum r_i a_i$, then $f < \sum r_i a_i$.

Lemma 21. *If* $g < \sum r_i a_i$ *and* $\sum r_i a_i \leqslant f$, *then* $g \leqslant f$.

Proof. Suppose that $1 = \bigvee_I ((g < r_I) \wedge a_I)$ and $a_I \leqslant (r_I < f)$. Then

$$1 \leqslant \bigvee_I ((g < r_I) \wedge (r_I < f)) \leqslant (0 < f - g).$$ □

Valuations on the lattice extend linearly to the (positive) simple functions generated by the opens. One proves by induction that

$$\sum_k \mu(\bigvee_{|K|=k, K\subset I} a_K) = \sum_{i\in I} \mu(a_i).$$

This is similar to the well-known principle of inclusion and exclusion. Consequently, $\sum_{i\in I} \mu(a_i) \leqslant \sum_{j\in J} \mu(b_j)$, when $\sum a_i \leqslant \sum b_j$. We extend the valuation to the formal complement $\overline{a}$ of an open a as the *upper* real $1 - \mu(a)$. Again the valuation extends to simple functions defined from formal complements of opens.

5.2. Approximable mappings.

Having completed the definition of the simple functions, we are now ready to show that the maps IV and VI defined above indeed map integrals to valuations, and vice versa, we need to check that the interpretations of all the axioms are valid.

Lemma 22. *IV is an approximable mapping*

Proof. We check that all the axioms of VAL are true under this interpretation.

V1. $[q<\mu(1)]=\top$ if $q<1$; $[q<\mu(\top)]=\bot$ if $q\geqslant 1$;
Since $[q<I(\frac{1+q}{2})]$ and $\frac{1+q}{2}<1$. The second part follows from **IP1**.

V2. $[q<\mu(0)]=\bot$ if $0\leqslant q$;
If $0\leqslant q$, then $\mathrm{IV}([q<\mu(0)])=\bigvee\{[q<I(g)]\colon g\leqslant 0\}=\bot$, because $g\leqslant q\cdot 1$.

V3. $[p<\mu(x)]\wedge[q<\mu(y)]\leqslant([a<\mu(x\vee y)])\vee([b<\mu(x\wedge y)])$ if $a+b=p+q$;
Suppose that $[p<I(f)]$, $f\leqslant x$, $[q<I(g)]$ and $g\leqslant y$. Since $f+g=f\vee g+f\wedge g$, we have $[a<I(f\vee g)]$ or $[b<I(f\wedge g)]$ by (I.3). Moreover, $f\vee g\leqslant x\vee y$ and $f\wedge g\leqslant x\wedge\ y$. Consequently, $\vdash\mathrm{IV}([a<\mu(x\vee y)])\vee\mathrm{IV}([b<\mu(x\wedge y)])$.
The proof for the other axiom is symmetric.

V4. $[p<\mu(x)]\leqslant[q<\mu(y)]$ if $x\leqslant y$ and $q\leqslant p$.
Suppose $[p<I(f)]$ and $f\leqslant x\leqslant y$, then $[q<I(f)]$ and $f\leqslant y$.

V5. Follows from the continuity axiom for integrals.

Reg. Follows from the continuity axiom for integrals. □

Lemma 23. *VI is an approximable mapping.*

Proof.

IP1. Is direct.

IP2/3. Let $f\in R$ and choose $a\leqslant f\leqslant b$. We consider the topos of sheaves over VAL. The map $\mu\mapsto\Delta_f(r,s):=\mu(r<f<s)$ is a sheaf which internally represents a Δ-function. More concretely, the interpretation of $\Delta_f(r,s)>q$ is the open $[q<\mu(D(f-r)\wedge D(s-f))]$. By the internal version of Proposition 16 $(\{p\mid p<\sum s_i\Delta(s_i,s_{i+1})\},\{q\mid\sum s_{i+1}\Delta[s_i,s_{i+1}]<q\})$ is a Dedekind real. In particular, if $s<t$, then

$$\bigvee(s_i).[\sum s_i\Delta_f(s_i,s_{i+1})>s]\vee\bigvee(s_i).[\sum s_{i+1}\Delta_f[s_i,s_{i+1}]<t]=\top.$$

Furthermore,

$$\bigvee(s_i).[\sum s_i\Delta_f(s_i,s_{i+1})>t]\vee\bigvee(s_i).[\sum s_{i+1}\Delta_f[s_i,s_{i+1}]<s]=\bot.$$

IP4. $[s+t<I(f+g)]\leqslant[s<I(f)]\vee[t<I(g)]$: Suppose that $[s+t<I(f+g)]$, then $[s+t+\varepsilon<\mu(a)]$ for some simple $a\leqslant f+g$. By **IP2** $[s<I(f)]\vee[I(f)<s+\varepsilon]=\top$, so it suffices to prove that $[I(f)<s+\varepsilon]\leqslant[t<I(g)]$. The interpretation of $[I(f)<s+\varepsilon]$ provides a simple $b=\sum r_i\overline{x_i}\geqslant f$ such that $[\mu(b)<s+\varepsilon]$. Let $M=\sum r_i$. Then $M-b=\sum r_ix_i$, $M-b\leqslant M-f$ and $[M-s-\varepsilon<\mu(M-b)]$. Moreover, $M-b+a\leqslant g+M$ and $[M+t<\mu(M-b+a)]$. This is the interpretation of $[t<I(g)]$.

IP5/6. The measures of the sets $(s<f\leqslant t)$, $(s<f<t)$ and $(s\leqslant f\leqslant t)$ are close when $\Delta[s]$ and $\Delta[t]$ are small. So the result follows from the observation that $\sum r_i(r_i<f\leqslant r_{i+1})=\sum(r_{i+1}-r_i)(r_{i+1}<f)$. □

5.3. Homeomorphism.

We prove that there is a homeomorphism between the integrals on a Riesz space and the valuations on the opens of the spectrum.

Theorem 24. [Riesz representation theorem] *Let R be a Riesz space with a strong unit. The theory of valuations on its spectrum is equivalent to the theory of integrals on R. It follows that the corresponding compact regular locales are homeomorphic.*

Proof. That is, we claim that $\mathrm{IV} \circ \mathrm{VI} = \mathrm{VI} \circ \mathrm{IV} = \mathrm{id}$.

Since the theories INT and INTPOS are equivalent, it suffices to consider only the latter.

We compute $\mathrm{IV} \circ \mathrm{VI}([p < I(f)])$. First, $\mathrm{VI}([p < I(f)]) = \bigvee (s_i).p < \sum s_i \Delta_f(s_i, s_{i+1})$. So, there are p_i such that $p < \sum s_i p_i$ and $p_i < \Delta_f(s_i, s_{i+1})$. By applying IV for each i, there exists g_i such that $p_i < s_i I(g_i)$ and $g_i \leqslant (s_i < f < s_{i+1})$.

We prove that for all f, $\mathrm{IV} \circ \mathrm{VI}\ ([p < I(f)]) \leqslant [p < I(f)]$. It follows from the computation above that $[p < I(\sum s_i g_i)]$. Since $\sum s_i g_i \leqslant f$ we have that $[p < I(f)]$.

Conversely, we claim that $[p < I(f)] \leqslant \mathrm{IV} \circ \mathrm{VI}\ ([p < I(f)])$. From $[p < I(f)]$ it follows that $[p + \varepsilon < I(f)]$ for some ε. We make a new partition (s_j) by adding two points a, b around every existing point such that $\Delta[a, b] \leqslant \frac{\varepsilon}{n}$. Consider $h_i := \frac{1}{s_{i+1} - s_i}(f \wedge s_{i+1} - f \wedge s_i)$. Then $(s_{i+1} < f) \leqslant h_i < (s_i \leqslant f)$ and $h_{i+i} \leqslant h_i$. Define $g_i := h_i - h_{i+1}$. Then $g_i < (s_i < f \leqslant s_{i+2})$ and $\sum s_i g_i = \sum (s_{i+1} - s_i) h_i = f$. Observe that the odd numbered gs live in the old intervals and that the even numbered functions cross these borders. By removing the last ones we obtain the requested sequence of functions.

We compute $\mathrm{VI} \circ \mathrm{IV}([p < \mu(u)])$. First apply IV, there exists g such that $g \leqslant 1$ and $D(g) \leqslant u$ and $[p < I(g)]$. Then, by VI, there exists a simple $y := \sum s_i (s_i < g < s_{i+1})$ such that $[p < \mu(y)]$.

We claim that $\mathrm{VI} \circ \mathrm{IV}\ ([p < \mu(u)]) \vdash [p < \mu(u)]$. By the computation above $y \leqslant D(g) \leqslant u$, hence $[p < \mu(u)]$.

Conversely, we claim that $[p < \mu(u)] \vdash \mathrm{VI} \circ \mathrm{IV}([p < \mu(u)])$. We may assume that $u = D(a)$, $0 \leqslant a \leqslant 1$. By regularity of the valuation we choose $D(a - \varepsilon)$ such that $[p < \mu(D(a - \varepsilon))]$. Choosing $g := \frac{1}{\varepsilon} a \wedge \varepsilon$ finishes the proof. □

6. CONCLUSIONS

The present construction was motivated by Bishop's bijection between measures and integrals [BB85]. Bishop's forces the measure of a measurable set to be computable. This is somewhat inconvenient in practice since for a measurable function f the measure of $[f \geqslant s]$ need not be computable in general. We believe that the present theory allows for a smoother development of, at least, the abstract functional analytic aspects of Bishop's measure theory.

This homeomorphism has already been applied in a *non-commutative* context of quantum theory [HLS08] where it provides an isomorphism between quasi-states and certain valuations. Quasi-states are used in the algebraic foundations of quantum mechanics.

Jackson [Jac] recently showed that a σ-algebra naturally carries a locale structure with the countable join topology and that the measures on the σ-algebra coincide with the continuous valuations on this locale. As such, continuous valuations on locales form a generalization of a measures on σ-algebras.

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