2IF55 Semantics and computational models

Recursion—transition systems and higher-order definitions

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Outline

J.W. de Bakker & E.P. de Vink, *Control Flow Semantics*
The MIT Press 1996

- Chapter 1: Recursion and Iterations
- Chapter 2: Nondeterminacy
- Chapter 4: Uniform Parallelism
- Chapter 11: Branching Domains at Work

§1.1.1 Syntax and operational semantics

a language with recursion

actions $a \in \mathsf{Act}$, recursion variables $x \in \mathsf{PVar}$

statements $s \in \mathsf{Stat}$ $s ::= a \mid x \mid (s;s)$

guarded statements $g \in \mathsf{GStat}$ $g ::= a \mid (g;s)$

declarations $D \in \mathsf{Decl} = \mathsf{PVar} \to \mathsf{GStat}$

programs $\pi \in \mathcal{L}_{rec} = \mathsf{Decl} \times \mathsf{Stat}$

computation sequences

transition system $\mathcal{T} = (\,\mathsf{Conf},\ \mathsf{Obs}, \rightarrow\,)$

configurations $c \in \mathsf{Conf}$

observations $a \in \mathsf{Obs}$

transitions $\rightarrow \subseteq \mathsf{Conf} \times \mathsf{Obs} \times \mathsf{Conf}$

computation sequence $c_0 \xrightarrow{a_1} c_1 \xrightarrow{a_2} c_2 \rightarrow \ldots \xrightarrow{a_n} c_n \rightarrow \ldots$
observable behaviour $a_1 \cdot a_2 \cdot \ldots \cdot a_n \cdot \ldots$

successor set $\mathcal{S}(c) = \{\,\langle\, a, c' \,\rangle \mid c \xrightarrow{a} c'\,\}$

no transtion $c \not\rightarrow$

transition system specification

(1) axiom $c \xrightarrow{a} c'$

(2) rule $\dfrac{c_1 \xrightarrow{a_1} c_1', \ldots, c_k \xrightarrow{a_k} c_k'}{c \xrightarrow{a} c'}$

(3) $\rightarrow$ is least set satisfying (1) and (2)

example 1

$$0 \xrightarrow{\leq} s(n) \tag{Ax}$$

$$\frac{n \xrightarrow{\leq} m}{s(n) \xrightarrow{\leq} s(m)} \tag{R1}$$

$$\frac{n \xrightarrow{\leq} m, \quad m \xrightarrow{\leq} p}{n \xrightarrow{\leq} p} \tag{R2}$$

example 2

$$q \xrightarrow{\varepsilon} q \qquad \text{(Ax1)}$$

$$q \xrightarrow{a} q' \qquad \text{if } q' \in \delta(q, a) \qquad \text{(Ax2)}$$

$$\frac{q \xrightarrow{a} q', \;\; q' \xrightarrow{w} q''}{q \xrightarrow{aw} q''} \qquad \text{(R)}$$

transition system specification $\mathcal{T}_{rec}$

resumption $\quad r \in \text{Res} \quad r ::= E \mid s$

$\mathcal{T}_{rec} = (\,\text{Decl} \times \text{Res},\ \text{Act},\ \rightarrow,\ \text{Spec}\,)$ with transitions $r_1 \xrightarrow{a}_D r_2$

$$a \xrightarrow{a}_D E \qquad \text{(Act)}$$

$$\frac{g \xrightarrow{a}_D r}{x \xrightarrow{a}_D} \quad \text{if } D(x) = g \qquad \text{(Rec)}$$

$$\frac{s_1 \xrightarrow{a}_D r_1}{s_1;s_2 \xrightarrow{a}_D r_1;s_2} \qquad \text{(Seq)}$$

lemma 1.8 $\exists r_1[\,(s_1;s_2);s_3 \xrightarrow{a}_D r_1\,] \iff \exists r_2[\,s_1(s_2;s_3) \xrightarrow{a}_D r_2\,]$

weights

weight function $\mathsf{wgt}\colon \mathsf{Decl} \times \mathsf{Res} \to \mathbb{N}$

$$
\begin{aligned}
\mathsf{wgt}(D|E) &= 0 \\
\mathsf{wgt}(D|a) &= 1 \\
\mathsf{wgt}(D|x) &= \mathsf{wgt}(D|D(x)) + 1 \\
\mathsf{wgt}(D|s_1; s_2) &= \mathsf{wgt}(D|s_1) + 1
\end{aligned}
$$

lemma 1.11 wgt is well-defined

lemma 1.9 $\mathcal{S}(D|S)$ has exactly one element finite and infinite words $\mathbb{P}_O = \mathsf{Act}^\infty = \mathsf{Act}^* \cup \mathsf{Act}^\omega$

operational semantics $\mathcal{O}_d \colon \mathsf{Decl} \times \mathsf{Res} \to \mathbb{P}_O$

$$\mathcal{O}_d(D|r) = \begin{cases} a_1 a_2 \cdots a_n & \text{if } r \xrightarrow{a_1}_D r_1 \xrightarrow{a_2}_D \ldots \xrightarrow{a_n}_D r_n \equiv E \\ a_1 a_2 \cdots & \text{if } r \xrightarrow{a_1}_D r_1 \xrightarrow{a_2}_D \ldots \end{cases}$$

lemma 1.14

$\mathcal{O}_d(D|x) = \mathcal{O}_d(D|D(x))$ and $\mathcal{O}_d(D|(s_1; s_2); s_3) = \mathcal{O}_d(D|s_1; (s_2; s_3))$

metric space (M, d) with set M and distance $d\colon M \times M \to \mathbb{R}_{\geq 0}$

$$d(x,y) = 0 \iff x = y \tag{M1}$$

$$d(x,y) = d(y,x) \tag{M2}$$

$$d(x,y) \leq d(x,z) + d(z,y) \tag{M3}$$

metric spaces

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ultrametric space (M, d) also satisfies

$$d(x,y) \leq \max\{\, d(x,z),\ d(z,y) \,\} \tag{M4}$$

Baire distance on A^∞

truncation $w[n]$

$$\begin{array}{rcl} w[0] & = & \epsilon \\ \epsilon[n+1] & = & \epsilon \\ (a \cdot v)[n+1] & = & a \cdot (\, v[n] \,) \end{array}$$

Baire distance $d_B(v,w) = \begin{cases} 0 & \text{if } v = w \\ 2^n & \text{if } v[n] = w[n] \text{ and } v[n+1] \neq w[n+1] \end{cases}$

lemma 1.21

- $d_B(v,w) \leq 2^{-n} \iff v[n] = w[n]$
- $d_B(a \cdot v, a \cdot w) = \frac{1}{2} d_B(v,w)$

composition of metric spaces

product space $M_1 \times M_2 = \{\, (x, y) \mid x \in M_1,\ y \in M_2 \,\}$

$$d_P((x, y), (x', y')) = \max\{\, d_1(x, x'),\ d_2(y, y') \,\}$$

disjoint union $M_1 + M_2 = (\{1\} \times M_1) \cup (\{2\} \times M_2)$

$$d_U(x, y) = \begin{cases} d_1(x', y') & \text{if } x = (1, x'),\ y = (1, y') \\ d_2(x', y') & \text{if } x = (2, x'),\ y = (2, y') \\ 1 & \text{otherwise} \end{cases}$$

function space $X \to M$ for *1-bounded* M

$$d_F(f, g) = \sup\{\, d(\, f(x), g(x)\,) \mid x \in X \,\}$$

lemma 1.24

- if M_1 and M_2 1-bounded, then also $M_1 \times M_2$ and $M_1 + M_2$
- if M_1 and M_2 ultrametric, then also $M_1 \times M_2$ and $M_1 + M_2$
- if M 1-bounded, then also $X \to F$
- if M ultrametric, then also $X \to F$

complete metric space

sequence $(x_n)_n$ convergent to limit x iff

$$\forall \varepsilon > 0 \exists i \forall j \geq i \colon d(x_j, x) \leq \varepsilon$$

sequence $(x_n)_n$ Cauchy iff

$$\forall \varepsilon > 0 \exists i \forall j, k \geq i \colon d(x_j, x_k) \leq \varepsilon$$

metric space M complete if

every Cauchy sequence converges to limit in M

lemma 1.28

if M_1, M_2, M complete then also $M_1 \times M_2, M_1 + M_2, X \to M$

an example of a complete space

theorem 1.29

(A^∞, d_B) is a 1-bounded complete ultrametric space