2IF55 Semantics and computational models

Recursion—transition systems and higher-order definitions

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Outline

J.W. de Bakker & E.P. de Vink, *Control Flow Semantics*
The MIT Press 1996

- Chapter 1: Recursion and Iterations
  Section 1.1: Recursion
- Chapter 2: Nondeterminacy
- Chapter 4: Uniform Parallelism
- Chapter 11: Branching Domains at Work

metric space (M, d) with set M and distance $d \colon M \times M \to \mathbb{R}_{\geq 0}$

$$d(x,y) = 0 \iff x = y \tag{M1}$$

$$d(x,y) = d(y,x) \tag{M2}$$

$$d(x,y) \leq d(x,z) + d(z,y) \tag{M3}$$

metric spaces

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ultrametric space (M, d) also satisfies

$$d(x, y) \leq \max\{\, d(x, z),\ d(z, y)\,\} \tag{M4}$$

Baire distance on A^∞

truncation $w[n]$

$$\begin{array}{rcl} w[0] & = & \epsilon \\ \epsilon[n+1] & = & \epsilon \\ (a \cdot v)[n+1] & = & a \cdot (\, v[n] \,) \end{array}$$

Baire distance $d_B(v, w) = \begin{cases} 0 & \text{if } v = w \\ 2^{-n} & \text{if } v[n] = w[n] \text{ and } v[n+1] \neq w[n+1] \end{cases}$

lemma 1.21

- $d_B(v, w) \leq 2^{-n} \iff v[n] = w[n]$
- $d_B(a \cdot v, a \cdot w) = \frac{1}{2} d_B(v, w)$

composition of metric spaces

product space $M_1 \times M_2 = \{\, (x, y) \mid x \in M_1,\ y \in M_2 \,\}$

$$d_P((x, y), (x', y')) = \max\{\, d_1(x, x'),\ d_2(y, y') \,\}$$

disjoint union $M_1 + M_2 = (\{1\} \times M_1) \cup (\{2\} \times M_2)$

$$d_U(x, y) = \begin{cases} d_1(x', y') & \text{if } x = (1, x'),\ y = (1, y') \\ d_2(x', y') & \text{if } x = (2, x'),\ y = (2, y') \\ 1 & \text{otherwise} \end{cases}$$

function space $X \to M$ for *1-bounded* M

$$d_F(f, g) = \sup\{\, d(\, f(x), g(x)\,) \mid x \in X \,\}$$

lemma 1.24

- if M_1 and M_2 1-bounded, then also $M_1 \times M_2$ and $M_1 + M_2$
- if M_1 and M_2 ultrametric, then also $M_1 \times M_2$ and $M_1 + M_2$
- if M 1-bounded, then also $X \to M$
- if M ultrametric, then also $X \to M$

complete metric space

sequence $(x_n)_n$ convergent to limit x iff

$$\forall \varepsilon > 0 \exists i \forall j \geq i \colon d(x_j, x) \leq \varepsilon$$

sequence $(x_n)_n$ Cauchy iff $\forall \varepsilon > 0 \exists i \forall j, k \geq i \colon d(x_j, x_k) \leq \varepsilon$

metric space M complete if

every Cauchy sequence converges to limit in M

lemma 1.28

if M_1, M_2, M complete then also $M_1 \times M_2, M_1 + M_2, X \to M$

theorem 1.29 (A^∞, d_B) 1-bounded complete ultrametric space

non-expansive and contractive functions

M_1, M_2 two metric spaces, $f \colon M_1 \to M_2$

f is non-expansive iff $\forall x, y \in M_1 \colon d_2(f(x), f(y)) \leq d_1(x, y)$

for $\alpha < 1$, f is α-contractive iff

$$\forall x, y \in M_1 \colon d_2(f(x), f(y)) \leq \alpha \cdot d_1(x, y)$$

notation $M_1 \xrightarrow{1} M_2$ and $M_1 \xrightarrow{\alpha} M_2$

if M_2 is complete, then so are $M_1 \xrightarrow{1} M_2$ and $M_1 \xrightarrow{\alpha} M_2$

Banach's theorem

complete metric space M, $f : M \to M$ contractive

- f has a fixed point: $\exists x \in M : x = f(x)$
- f has at most one fixed point:
  $\forall x, y \in M : (\, x = f(x) \wedge y = f(y) \,) \Rightarrow x = y$
- for any $x_0 \in M$, if $x = \lim_n f^n(x_0)$ then $x = f(x)$

notation $\mathrm{fix}(f)$