2IF55 Semantics and computational models

Iteration—a non-uniform language with continuation semantics

Technische Universiteit Eindhoven

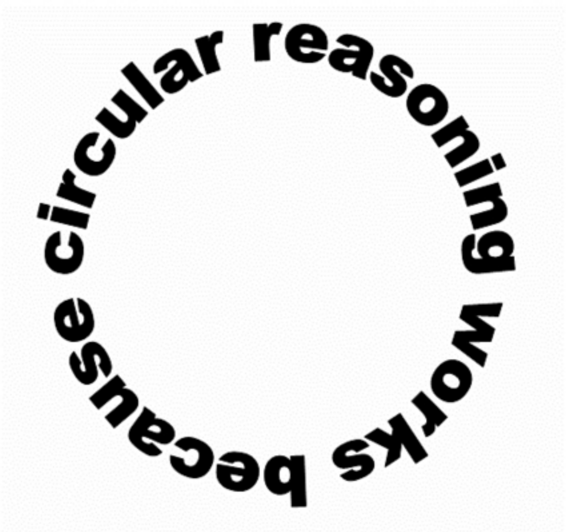
December 15, 2009

Outline

J.W. de Bakker & E.P. de Vink, *Control Flow Semantics*
The MIT Press 1996

- Chapter 1: Recursion and Iterations
  Section 1.1: Recursion
  Section 1.2: Iteration

fixed point construction



§1.2.1 Syntax and operational semantics

syntax of $\mathcal{L}_{wh}$

$v \in \text{IVar}$ individual variables $\quad e \in \text{Exp}$ expressions

$$s \in \text{Stat} ::= v := e \mid \textbf{skip} \mid s; s \mid$$
$$\textbf{if } e \textbf{ then } s \textbf{ else } s \textbf{ fi} \mid \textbf{while } e \textbf{ do } s \textbf{ od}$$

$$\pi \in \mathcal{L}_{wh} = \text{Stat}$$

syntax of $\mathcal{L}_{wh}$

$v \in \mathrm{IVar}$ individual variables $\qquad e \in \mathrm{Exp}$ expressions

$$s \in \mathrm{Stat} ::= v := e \mid \textbf{skip} \mid s; s \mid$$
$$\textbf{if } e \textbf{ then } s \textbf{ else } s \textbf{ fi} \mid \textbf{while } e \textbf{ do } s \textbf{ od}$$

$$\pi \in \mathcal{L}_{wh} = \mathrm{Stat}$$

characteristic equivalence

$$\textbf{while } e \textbf{ do } s \textbf{ od} = \textbf{if } e \textbf{ then } (s; \textbf{while } e \textbf{ do } s \textbf{ od}) \textbf{ else skip fi}$$

states in Σ and valuation $\mathcal{V}$

$\alpha \in \mathsf{Val}$ values $\mathsf{tt}, \mathsf{ff} \in \mathsf{Val}$

$\sigma \in \Sigma = \mathsf{IVar} \to \mathsf{Val}$ states

variant of σ with α for v: $\sigma\{\alpha/v\}(v') = \begin{cases} \alpha & \text{if } v = v' \\ \sigma(v') & \text{if } v \neq v' \end{cases}$

$\mathcal{V}\colon \mathsf{Exp} \to \Sigma \to \mathsf{Val}$ evaluation function

zero-step derivation

$r \in Res ::= E \mid s : r$ resumptions

$[r_1, \sigma_1] \rightarrow_0 [r_2, \sigma_2]$ short for $\dfrac{[r_2, \sigma_2] \rightarrow [r, \sigma]}{[r_1, \sigma_1] \rightarrow [r, \sigma]}$

transition system $\mathcal{T}_{wh}$

- $[(v := e) : r, \sigma] \to [r, \sigma\{\alpha/v\}]$ where $\alpha = \mathcal{V}(e)(\sigma)$
- $[\textbf{skip} : r, \sigma] \to [r, \sigma]$
- $[(s_1; s_2) : r, \sigma] \to_0 [s_1 : (s_2 : r), \sigma]$
- $[\textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi} : r, \sigma] \to_0 [s_1 : r, \sigma]$ if $\mathcal{V}(e)(\sigma) = \text{tt}$
- $[\textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi} : r, \sigma] \to_0 [s_2 : r, \sigma]$ if $\mathcal{V}(e)(\sigma) = \text{ff}$
- $[\textbf{while } e \textbf{ do } s \textbf{ od} : r, \sigma] \to_0$

  $[\textbf{if } e \textbf{ then } (s; \textbf{while } e \textbf{ do } s \textbf{ od}) \textbf{ else skip fi} : r, \sigma]$

fixed point transformation $\Phi\colon \mathrm{Sem}_O \to \mathrm{Sem}_O$
where $\mathrm{Sem}_O = \mathrm{Res} \times \Sigma \to \Sigma^\infty$

$$\begin{array}{rcll} \Phi(S)(E) & = & \epsilon & \\ \Phi(S)(s : r, \sigma) & = & \sigma' \cdot S(r', \sigma') & \quad \text{if } [s : r, \sigma] \to [r', \sigma'] \end{array}$$

operational semantics $\mathcal{O}$ and $\mathcal{O}[\![\cdot]\!]$

$$\begin{array}{rcl} \mathcal{O}\colon \mathrm{Res} \times \Sigma \to \Sigma^\infty & \text{given by} & \mathcal{O} = \mathrm{fix}(\Phi) \\ \mathcal{O}[\![\cdot]\!]\colon \mathcal{L}_{wh} \to \Sigma \to \Sigma^\infty & \text{given by} & \mathcal{O}[\![s]\!](\sigma) = \mathcal{O}(s : E, \sigma) \end{array}$$

well-definedness and contractivity of Φ

weight wgt on Res and Stat

$$
\begin{aligned}
\mathrm{wgt}(E) &= 0 \\
\mathrm{wgt}(s : r) &= \mathrm{wgt}(s) \\
\mathrm{wgt}(v := e) &= 1 \\
\mathrm{wgt}(\mathbf{skip}) &= 1 \\
\mathrm{wgt}(s_1 ; s_2) &= \mathrm{wgt}(s_1) + 1 \\
\mathrm{wgt}(\mathbf{if}\ e\ \mathbf{then}\ s_1\ \mathbf{else}\ s_2\ \mathbf{fi}) &= \max\{\, \mathrm{wgt}(s_1), \mathrm{wgt}(s_2) \,\} + 1 \\
\mathrm{wgt}(\mathbf{while}\ e\ \mathbf{do}\ s\ \mathbf{od}) &= \\
&\mathrm{wgt}(\mathbf{if}\ e\ \mathbf{then}\ (s; \mathbf{while}\ e\ \mathbf{do}\ s\ \mathbf{od})\ \mathbf{else}\ \mathbf{skip}\ \mathbf{fi}) + 1
\end{aligned}
$$

lemma 1.61 Φ is $\frac{1}{2}$-contractive

well-definedness and contractivity of Φ

weight wgt on Res and Stat

$$
\begin{array}{rcl}
\mathrm{wgt}(E) & = & 0 \\
\mathrm{wgt}(s : r) & = & \mathrm{wgt}(s) \\
\mathrm{wgt}(v := e) & = & 1 \\
\mathrm{wgt}(\mathbf{skip}) & = & 1 \\
\mathrm{wgt}(s_1; s_2) & = & \boxed{\mathrm{wgt}(s_1) + 1} \\
\mathrm{wgt}(\mathbf{if}\ e\ \mathbf{then}\ s_1\ \mathbf{else}\ s_2\ \mathbf{fi}) & = & \max\{\, \mathrm{wgt}(s_1), \mathrm{wgt}(s_2) \,\} + 1 \\
\mathrm{wgt}(\mathbf{while}\ e\ \mathbf{do}\ s\ \mathbf{od}) & = &
\end{array}
$$

$$
\mathrm{wgt}(\mathbf{if}\ e\ \mathbf{then}\ (s; \mathbf{while}\ e\ \mathbf{do}\ s\ \mathbf{od})\ \mathbf{else}\ \mathbf{skip}\ \mathbf{fi}) + 1
$$

lemma 1.61 Φ is $\frac{1}{2}$-contractive

well-definedness and contractivity of Φ

weight wgt on Res and Stat

$$
\begin{aligned}
\mathrm{wgt}(E) &= 0 \\
\mathrm{wgt}(s : r) &= \mathrm{wgt}(s) \\
\mathrm{wgt}(v := e) &= 1 \\
\mathrm{wgt}(\textbf{skip}) &= 1 \\
\mathrm{wgt}(s_1 ; s_2) &= \mathrm{wgt}(s_1) + 1 \\
\mathrm{wgt}(\textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi}) &= \boxed{\max\{\, \mathrm{wgt}(s_1), \mathrm{wgt}(s_2) \,\} + 1} \\
\mathrm{wgt}(\textbf{while } e \textbf{ do } s \textbf{ od}) &= \\
&\mathrm{wgt}(\textbf{if } e \textbf{ then } (s; \textbf{while } e \textbf{ do } s \textbf{ od}) \textbf{ else skip fi}) + 1
\end{aligned}
$$

lemma 1.61 Φ is $\frac{1}{2}$-contractive weight wgt on Res and Stat

$$
\begin{array}{rcl}
\mathrm{wgt}(E) & = & 0 \\
\mathrm{wgt}(s : r) & = & \mathrm{wgt}(s) \\
\mathrm{wgt}(v := e) & = & 1 \\
\mathrm{wgt}(\mathbf{skip}) & = & 1 \\
\mathrm{wgt}(s_1 ; s_2) & = & \mathrm{wgt}(s_1) + 1 \\
\mathrm{wgt}(\mathbf{if}\ e\ \mathbf{then}\ s_1\ \mathbf{else}\ s_2\ \mathbf{fi}) & = & \max\{\, \mathrm{wgt}(s_1), \mathrm{wgt}(s_2)\,\} + 1 \\
\mathrm{wgt}(\mathbf{while}\ e\ \mathbf{do}\ s\ \mathbf{od}) & = & \\
\end{array}
$$

$$
\boxed{\mathrm{wgt}(\mathbf{if}\ e\ \mathbf{then}\ (s; \mathbf{while}\ e\ \mathbf{do}\ s\ \mathbf{od})\ \mathbf{else\ skip\ fi}) + 1}
$$

lemma 1.61 Φ is $\frac{1}{2}$-contractive

continuation semantics for $\mathcal{L}_{wh}$

$\gamma \in \text{Cont} = \Sigma \to \Sigma^\infty$ continuations

empty continuation γ_ϵ such that $\gamma_\epsilon(\sigma) = \epsilon$

denotational semantics $\mathcal{D} \colon \mathcal{L}_{wh} \to \text{Cont} \to \Sigma \to \Sigma^\infty$

$$\mathcal{D}(v := e)(\gamma)(\sigma) =$$

$$\mathcal{D}(\textbf{skip})(\gamma)(\sigma) =$$

$$\mathcal{D}(s_1; s_2)(\gamma)(\sigma) =$$

$$\mathcal{D}(\textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi})(\gamma)(\sigma) =$$

$$\mathcal{D}(\textbf{while } e \textbf{ do } s \textbf{ od})(\gamma)(\sigma) =$$

$\mathcal{D}[\![\cdot]\!] \colon \mathcal{L}_{wh} \to \Sigma \to \Sigma^\infty$ given by $\mathcal{D}[\![s]\!] = \mathcal{D}(s)(\gamma_\epsilon)$

$\gamma \in \mathsf{Cont} = \Sigma \to \Sigma^\infty$ continuations

empty continuation γ_ϵ such that $\gamma_\epsilon(\sigma) = \epsilon$

denotational semantics $\mathcal{D} \colon \mathcal{L}_{wh} \to \mathsf{Cont} \to \Sigma \to \Sigma^\infty$

$$\mathcal{D}(v := e)(\gamma)(\sigma) = \sigma\{\alpha/v\} \cdot \gamma(\sigma\{\alpha/v\})$$

where $\alpha = \mathcal{V}(e)(\sigma)$

$$\mathcal{D}(\textbf{skip})(\gamma)(\sigma) =$$

$$\mathcal{D}(s_1; s_2)(\gamma)(\sigma) =$$

$$\mathcal{D}(\textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi})(\gamma)(\sigma) =$$

$$\mathcal{D}(\textbf{while } e \textbf{ do } s \textbf{ od})(\gamma)(\sigma) =$$

$\mathcal{D}[\![\cdot]\!] \colon \mathcal{L}_{wh} \to \Sigma \to \Sigma^\infty$ given by $\mathcal{D}[\![s]\!] = \mathcal{D}(s)(\gamma_\epsilon)$

continuation semantics for $\mathcal{L}_{wh}$

$\gamma \in \mathsf{Cont} = \Sigma \to \Sigma^\infty$ continuations

empty continuation γ_ϵ such that $\gamma_\epsilon(\sigma) = \epsilon$

denotational semantics $\mathcal{D} \colon \mathcal{L}_{wh} \to \mathsf{Cont} \to \Sigma \to \Sigma^\infty$

$$\begin{aligned}
\mathcal{D}(v := e)(\gamma)(\sigma) &= \sigma\{\alpha/v\} \cdot \gamma(\sigma\{\alpha/v\}) \\
&\qquad \text{where } \alpha = \mathcal{V}(e)(\sigma) \\
\mathcal{D}(\mathbf{skip})(\gamma)(\sigma) &= \sigma \cdot \gamma(\sigma) \\
\mathcal{D}(s_1; s_2)(\gamma)(\sigma) &= \\
\mathcal{D}(\mathbf{if}\ e\ \mathbf{then}\ s_1\ \mathbf{else}\ s_2\ \mathbf{fi})(\gamma)(\sigma) &= \\
\mathcal{D}(\mathbf{while}\ e\ \mathbf{do}\ s\ \mathbf{od})(\gamma)(\sigma) &=
\end{aligned}$$

$\mathcal{D}[\![\cdot]\!] \colon \mathcal{L}_{wh} \to \Sigma \to \Sigma^\infty$ given by $\mathcal{D}[\![s]\!] = \mathcal{D}(s)(\gamma_\epsilon)$

$\gamma \in \mathsf{Cont} = \Sigma \to \Sigma^\infty$ continuations

empty continuation γ_ϵ such that $\gamma_\epsilon(\sigma) = \epsilon$

denotational semantics $\mathcal{D} \colon \mathcal{L}_{wh} \to \mathsf{Cont} \to \Sigma \to \Sigma^\infty$

$$\begin{aligned}
\mathcal{D}(v := e)(\gamma)(\sigma) &= \sigma\{\alpha/v\} \cdot \gamma(\sigma\{\alpha/v\}) \\
&\qquad\qquad \text{where } \alpha = \mathcal{V}(e)(\sigma) \\
\mathcal{D}(\textbf{skip})(\gamma)(\sigma) &= \sigma \cdot \gamma(\sigma) \\
\mathcal{D}(s_1; s_2)(\gamma)(\sigma) &= \mathcal{D}(s_1)(\,\mathcal{D}(s_2)(\gamma)\,)(\sigma) \\
\mathcal{D}(\textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi})(\gamma)(\sigma) &= \\
\mathcal{D}(\textbf{while } e \textbf{ do } s \textbf{ od})(\gamma)(\sigma) &=
\end{aligned}$$

$\mathcal{D}[\![\cdot]\!] \colon \mathcal{L}_{wh} \to \Sigma \to \Sigma^\infty$ given by $\mathcal{D}[\![s]\!] = \mathcal{D}(s)(\gamma_\epsilon)$

continuation semantics for $\mathcal{L}_{wh}$

$\gamma \in \text{Cont} = \Sigma \to \Sigma^\infty$ continuations

empty continuation γ_ϵ such that $\gamma_\epsilon(\sigma) = \epsilon$

denotational semantics $\mathcal{D} \colon \mathcal{L}_{wh} \to \text{Cont} \to \Sigma \to \Sigma^\infty$

$$
\begin{aligned}
\mathcal{D}(v := e)(\gamma)(\sigma) &= \sigma\{\alpha/v\} \cdot \gamma(\sigma\{\alpha/v\}) \\
&\qquad\qquad \text{where } \alpha = \mathcal{V}(e)(\sigma) \\
\mathcal{D}(\textbf{skip})(\gamma)(\sigma) &= \sigma \cdot \gamma(\sigma) \\
\mathcal{D}(s_1; s_2)(\gamma)(\sigma) &= \mathcal{D}(s_1)(\,\mathcal{D}(s_2)(\gamma)\,)(\sigma) \\
\mathcal{D}(\textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi})(\gamma)(\sigma) &= \mathcal{D}(s_1)(\gamma)(\sigma) \qquad \text{if } \mathcal{V}(e)(\sigma) = \text{tt}
\end{aligned}
$$

$$
\mathcal{D}(\textbf{while } e \textbf{ do } s \textbf{ od})(\gamma)(\sigma) =
$$

$\mathcal{D}[\![\cdot]\!] \colon \mathcal{L}_{wh} \to \Sigma \to \Sigma^\infty$ given by $\mathcal{D}[\![s]\!] = \mathcal{D}(s)(\gamma_\epsilon)$

continuation semantics for $\mathcal{L}_{wh}$

$\gamma \in \text{Cont} = \Sigma \to \Sigma^\infty$ continuations

empty continuation γ_ϵ such that $\gamma_\epsilon(\sigma) = \epsilon$

denotational semantics $\mathcal{D}\colon \mathcal{L}_{wh} \to \text{Cont} \to \Sigma \to \Sigma^\infty$

$$
\begin{aligned}
\mathcal{D}(v := e)(\gamma)(\sigma) &= \sigma\{\alpha/v\} \cdot \gamma(\sigma\{\alpha/v\}) \\
&\qquad \text{where } \alpha = \mathcal{V}(e)(\sigma) \\
\mathcal{D}(\textbf{skip})(\gamma)(\sigma) &= \sigma \cdot \gamma(\sigma) \\
\mathcal{D}(s_1; s_2)(\gamma)(\sigma) &= \mathcal{D}(s_1)(\,\mathcal{D}(s_2)(\gamma)\,)(\sigma) \\
\mathcal{D}(\textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi})(\gamma)(\sigma) &= \mathcal{D}(s_1)(\gamma)(\sigma) \qquad \text{if } \mathcal{V}(e)(\sigma) = \text{tt} \\
&= \mathcal{D}(s_2)(\gamma)(\sigma) \qquad \text{if } \mathcal{V}(e)(\sigma) = \text{ff} \\
\mathcal{D}(\textbf{while } e \textbf{ do } s \textbf{ od})(\gamma)(\sigma) &=
\end{aligned}
$$

$\mathcal{D}[\![\cdot]\!]\colon \mathcal{L}_{wh} \to \Sigma \to \Sigma^\infty$ given by $\mathcal{D}[\![s]\!] = \mathcal{D}(s)(\gamma_\epsilon)$

continuation semantics for $\mathcal{L}_{wh}$

$\gamma \in \mathrm{Cont} = \Sigma \to \Sigma^\infty$ continuations

empty continuation γ_ϵ such that $\gamma_\epsilon(\sigma) = \epsilon$

denotational semantics $\mathcal{D} \colon \mathcal{L}_{wh} \to \mathrm{Cont} \to \Sigma \to \Sigma^\infty$

$$\begin{aligned}
\mathcal{D}(v := e)(\gamma)(\sigma) &= \sigma\{\alpha/v\} \cdot \gamma(\sigma\{\alpha/v\}) \\
&\qquad\qquad \text{where } \alpha = \mathcal{V}(e)(\sigma) \\
\mathcal{D}(\textbf{skip})(\gamma)(\sigma) &= \sigma \cdot \gamma(\sigma) \\
\mathcal{D}(s_1; s_2)(\gamma)(\sigma) &= \mathcal{D}(s_1)(\,\mathcal{D}(s_2)(\gamma)\,)(\sigma) \\
\mathcal{D}(\textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi})(\gamma)(\sigma) &= \mathcal{D}(s_1)(\gamma)(\sigma) \qquad \text{if } \mathcal{V}(e)(\sigma) = \mathrm{tt} \\
&= \mathcal{D}(s_2)(\gamma)(\sigma) \qquad \text{if } \mathcal{V}(e)(\sigma) = \mathrm{ff} \\
\mathcal{D}(\textbf{while } e \textbf{ do } s \textbf{ od})(\gamma)(\sigma) &= \\
\mathcal{D}(\textbf{if } e \textbf{ then } (s; \textbf{while } e \textbf{ do } s \textbf{ od}) &\textbf{ else skip fi})(\gamma)(\sigma)
\end{aligned}$$

$\mathcal{D}[\![\cdot]\!] \colon \mathcal{L}_{wh} \to \Sigma \to \Sigma^\infty$ given by $\mathcal{D}[\![s]\!] = \mathcal{D}(s)(\gamma_\epsilon)$

fixed point charaterization of $\mathcal{D}$

$\Psi \colon \mathrm{Sem}_D \to \mathrm{Sem}_D$ where $\mathrm{Sem}_D = \mathrm{Stat} \to \mathrm{Cont} \xrightarrow{1/2} \Sigma \to \Sigma^\infty$

$$
\begin{aligned}
\Psi(S)(v := e)(\gamma)(\sigma) &= \sigma\{\alpha/v\} \cdot \gamma(\sigma\{\alpha/v\}) \\
&\qquad\qquad \text{where } \alpha = \mathcal{V}(e)(\sigma) \\
\Psi(S)(\textbf{skip})(\gamma)(\sigma) &= \sigma \cdot \gamma(\sigma) \\
\Psi(S)(s_1; s_2)(\gamma)(\sigma) &= \Psi(S)(s_1)(\, S(s_2)(\gamma)\,)(\sigma) \\
\Psi(S)(\textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi})(\gamma)(\sigma) &= \Psi(S)(s_1)(\gamma)(\sigma) \qquad \text{if } \mathcal{V}(e)(\sigma) = \mathrm{tt} \\
&= \Psi(S)(s_2)(\gamma)(\sigma) \qquad \text{if } \mathcal{V}(e)(\sigma) = \mathrm{tt} \\
\Psi(S)(\textbf{while } e \textbf{ do } s \textbf{ od})(\gamma)(\sigma) &= \\
\Psi(S)(\textbf{if } e \textbf{ then } (s; \textbf{while } e \textbf{ do } s \textbf{ od}) &\textbf{ else skip fi})(\gamma)(\sigma)
\end{aligned}
$$

lemma 1.63 Ψ is $\frac{1}{2}$-contractive extended semantics $\mathcal{E} : \mathrm{Res} \to \Sigma \to \Sigma^\infty$

$$\begin{aligned} \mathcal{E}(E) &= \gamma_\epsilon \\ \mathcal{E}(s : r) &= \mathcal{D}(s)(\,\mathcal{E}(r)\,) \end{aligned}$$

extended semantics $\mathcal{E}^* : \mathrm{Res} \times \Sigma \to \Sigma^\infty$ such that

$$\mathcal{E}^*(r, \sigma) = \mathcal{E}(r)(\sigma)$$

lemma 1.65 $\Phi(\mathcal{E}^*) = \mathcal{E}^*$

theorem 1.67 $\mathcal{O}[\![\pi]\!] = \mathcal{D}[\![\pi]\!]$ for $\pi \in \mathcal{L}_{wh}$