Opgave 1

a) Zie Definitie 1.30 op blz. 40. Voor twee metrische ruimten (M_1, d_1), (M_2, d_2), een afbeelding $f\colon M_1 \to M_2$ is een $\frac{1}{2}$-contractie als $\forall x, y \in M_1\colon d_2(f(x), f(y)) \leq \frac{1}{2} d_1(x, y)$.

b) Kies $w, v \in A^\infty$. Er geldt $d_B(\mathit{pref}_c(w), \mathit{pref}_c(v)) = d_B(c \cdot w, c \cdot v) = \frac{1}{2} d_B(w, v)$. Dus pref_c is een $\frac{1}{2}$-contractie voor (A^∞, d_B).

Opgave 2

a)

$$\frac{\dfrac{a \xrightarrow{a}_D \mathrm{E}}{a; y \xrightarrow{a}_D y}}{x \xrightarrow{a}_D y} \qquad \frac{\dfrac{b \xrightarrow{b}_D \mathrm{E}}{b; y \xrightarrow{b}_D y}}{y \xrightarrow{b}_D y}$$

b) We hebben $d_B(\, \mathcal{O}(D|y), b^\omega) = d_B(b \cdot \mathcal{O}(D|y), b \cdot b^\omega) = \frac{1}{2} d_B(\mathcal{O}(D|y), b^\omega)$. Dus $d_B(\mathcal{O}(D|y), b^\omega) = 0$ en $\mathcal{O}(D|y) = b^\omega$. Hieruit volgt $d_B(\mathcal{O}(D|x), a \cdot b^\omega) = d_B(a \cdot \mathcal{O}(D|y), a \cdot b^\omega) = \frac{1}{2} d_B(\mathcal{O}(D|y), b^\omega) = 0$ en $\mathcal{O}(D|x) = a \cdot b^\omega$.

Opgave 3

a) Enerzijds $\mathcal{D}(s_1; (s_2; s_3))(\gamma)(\sigma) = \mathcal{D}(s_1)\Big(\, \mathcal{D}(s_2; s_3)(\gamma)\, \Big)(\sigma) = \mathcal{D}(s_1)\Big(\, \mathcal{D}(s_2)(\, \mathcal{D}(s_3)(\gamma)\,)\, \Big)(\sigma)$. Anderzijds $\mathcal{D}((s_1; s_2); s_3))(\gamma)(\sigma) = \mathcal{D}(s_1; s_2)(\, \mathcal{D}(s_3)(\gamma)\,)(\sigma) = \mathcal{D}(s_1)\Big(\, \mathcal{D}(s_2)(\, \mathcal{D}(s_3)(\gamma)\,)\, \Big)(\sigma)$. Dus $\mathcal{D}(s_1; (s_2; s_3))(\gamma)(\sigma) = \mathcal{D}((s_1; s_2); s_3))(\gamma)(\sigma)$.

b) Zet $\mathit{WHILE}_1 = \textbf{while } e \textbf{ do if } e \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi od})(\gamma)(\sigma)$ en $\mathit{WHILE}_2 = \textbf{while } e \textbf{ do } s_1 \textbf{ od}$. Kies γ en σ willekeurg. Stel $\mathcal{V}(e)(\sigma) = \mathit{tt}$. Dan,

$$\begin{aligned}
&\mathcal{D}(\mathit{WHILE}_1)(\gamma)(\sigma) \\
&\quad = \mathcal{D}(\textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi}; \mathit{WHILE}_1)(\gamma)(\sigma) \\
&\quad = \mathcal{D}(\textbf{if } e \textbf{ then } s_1 \textbf{ else } s_2 \textbf{ fi})(\, \mathcal{D}(\mathit{WHILE}_1)(\gamma)\,)(\sigma) \\
&\quad = \mathcal{D}(s_1)(\, \mathcal{D}(\mathit{WHILE}_1)(\gamma)\,)(\sigma)
\end{aligned}$$

en

$$\begin{aligned}
&\mathcal{D}(\mathit{WHILE}_2)(\gamma)(\sigma) \\
&\quad = \mathcal{D}(s_1; \mathit{WHILE}_2)(\gamma)(\sigma) \\
&\quad = \mathcal{D}(s_1)(\, \mathcal{D}(\mathit{WHILE}_2)(\gamma)\,)(\sigma).
\end{aligned}$$

Stel $\mathcal{V}(e)(\sigma) = \mathit{ff}$. Dan, $\mathcal{D}(\mathit{WHILE}_1)(\gamma)(\sigma) = \mathcal{D}(\textbf{skip})(\gamma)(\sigma) = \sigma \cdot \gamma(\sigma)$ en $\mathcal{D}(\mathit{WHILE}_2)(\gamma)(\sigma) = \mathcal{D}(\textbf{skip})(\gamma)(\sigma) = \sigma \cdot \gamma(\sigma)$. Hieruit volgt

$$\begin{aligned}
&d_B(\, \mathcal{D}(\mathit{WHILE}_1)(\gamma)(\sigma), \mathcal{D}(\mathit{WHILE}_2)(\gamma)(\sigma)\,) \\
&\quad \leq \max\{\, d_B(\, \mathcal{D}(s_1)(\, \mathcal{D}(\mathit{WHILE}_1)(\gamma)(\sigma), \mathcal{D}(s_1)(\, \mathcal{D}(\mathit{WHILE}_2)(\gamma)(\sigma)\,), \\
&\qquad\qquad d_B(\, \sigma \cdot \gamma(\sigma), \sigma \cdot \gamma(\sigma)\,)\,\} \\
&\quad = d_B(\, \mathcal{D}(s_1)(\, \mathcal{D}(\mathit{WHILE}_1)(\gamma)(\sigma), \mathcal{D}(s_1)(\, \mathcal{D}(\mathit{WHILE}_2)(\gamma)(\sigma)\,) \\
&\quad \leq \tfrac{1}{2} d_B(\, \mathcal{D}(\mathit{WHILE}_1)(\gamma), \mathcal{D}(\mathit{WHILE}_2)(\gamma)\,)
\end{aligned}$$

want $\mathcal{D}(s_1)$ is $\frac{1}{2}$-contraherend in γ. Dus

$$\begin{aligned}
&d_B(\, \mathcal{D}(\mathit{WHILE}_1)(\gamma), \mathcal{D}(\mathit{WHILE}_2)(\gamma)\,) \leq \tfrac{1}{2} d_B(\, \mathcal{D}(\mathit{WHILE}_1)(\gamma), \mathcal{D}(\mathit{WHILE}_2)(\gamma)\,), \\
&d_B(\, \mathcal{D}(\mathit{WHILE}_1)(\gamma), \mathcal{D}(\mathit{WHILE}_2)(\gamma)\,) = 0 \quad \text{en} \\
&\mathcal{D}(\mathit{WHILE}_1)(\gamma) = \mathcal{D}(\mathit{WHILE}_2)(\gamma).
\end{aligned}$$

Conclusie $\mathcal{D}(\mathit{WHILE}_1)(\gamma)(\sigma) = \mathcal{D}(\mathit{WHILE}_2)(\gamma)(\sigma)$.

Opgave 4 Er geldt, wegens de definities, $\Phi(\mathcal{E})(D|\mathrm{E}) = \epsilon$ en $\mathcal{E}(D|\mathrm{E}) = \epsilon$. Dus $\Phi(\mathcal{E})(D|\mathrm{E}) = \mathcal{E}(D|\mathrm{E})$. Zie Lemma 1.49 op blz. 55 voor een uitwerking van het bewijs dat $\Phi(\mathcal{E})(D|s) = \mathcal{E}(D|s)$ met inductie naar $\mathit{wgt}(s)$. Conclusie $\Phi(\mathcal{E})(D|r) = \mathcal{E}(D|r)$ voor alle $r \in \mathit{Res}$, $\Phi(\mathcal{E}) = \mathcal{E}$ en $\mathcal{E}$ is een dekpunt van Φ.