Theorem 3.5.3 Let x, y, z be three terms over $SA_\tau(\mathcal{N})$. Then $(x \cdot y) \cdot z \leftrightarrows x \cdot (y \cdot z)$.

Proof Put $R = \{\, (\, (x' \cdot y) \cdot z,\ x' \cdot (y \cdot z)\,) \mid x' \in SA_\tau(\mathcal{N})\,\} \cup \Delta$. We verify that R is a bisimulation relation.

Suppose $(x' \cdot y) \cdot z \xrightarrow{a} p$. Then (i) $x' \xrightarrow{a} x''$ and $p = (x'' \cdot y) \cdot z$, or (ii) $x'{\downarrow}$, $y \xrightarrow{a} y'$ and $p = y' \cdot z$, or (iii) $x'{\downarrow}$, $y{\downarrow}$, $z \xrightarrow{a} z'$ and $p = z'$. Thus (i) $x' \cdot (y \cdot z) \xrightarrow{a} x'' \cdot (y \cdot z)$, or (ii) $x' \cdot (y \cdot z) \xrightarrow{a} y' \cdot z$, or (iii) $x' \cdot (y \cdot z) \xrightarrow{a} z'$. In all cases we can choose $q \in SA_\tau(\mathcal{N})$ such that $x' \cdot (y \cdot z) \xrightarrow{a} q$ and $R(p, q)$.

Reversely, suppose $x' \cdot (y \cdot z) \xrightarrow{a} q$. Then (i) $x' \xrightarrow{a} x''$ and $q = x'' \cdot (y \cdot z)$, or (ii) $x'{\downarrow}$, $y \xrightarrow{a} y'$ and $q = y' \cdot z$, or (iii) $x'{\downarrow}$, $y{\downarrow}$, $z \xrightarrow{a} z'$ and $q = z'$. Thus (i) $(x' \cdot y) \cdot z \xrightarrow{a} (x'' \cdot y) \cdot z$, or (ii) $(x' \cdot y) \cdot z \xrightarrow{a} y' \cdot z$, or (iii) $(x' \cdot y) \cdot z \xrightarrow{a} z'$. In all cases we can choose $pin SA_\tau(\mathcal{N})$ such that $(x' \cdot y) \cdot z \xrightarrow{a} p$ and $R(p, q)$.

As to termination we have $((x' \cdot y) \cdot z){\downarrow}$ iff $(x' \cdot y){\downarrow}$ and $z{\downarrow}$ iff $x'{\downarrow}$, $y{\downarrow}$ and $z{\downarrow}$ iff $x'{\downarrow}$ and $(y \cdot z){\downarrow}$ iff $(x' \cdot (y \cdot z)){\downarrow}$.

We conclude that R is a bisimulation relation. Since $R(\,(x \cdot y) \cdot z,\ x \cdot (y \cdot z)\,)$ we have $(x \cdot y) \cdot z \leftrightarrows x \cdot (y \cdot z)$.