- NP-completeness

  language C is NP-complete if

  - C is NP
  - $A \leq_P C$ for all A in NP

- Cook's theorem SAT is NP-complete

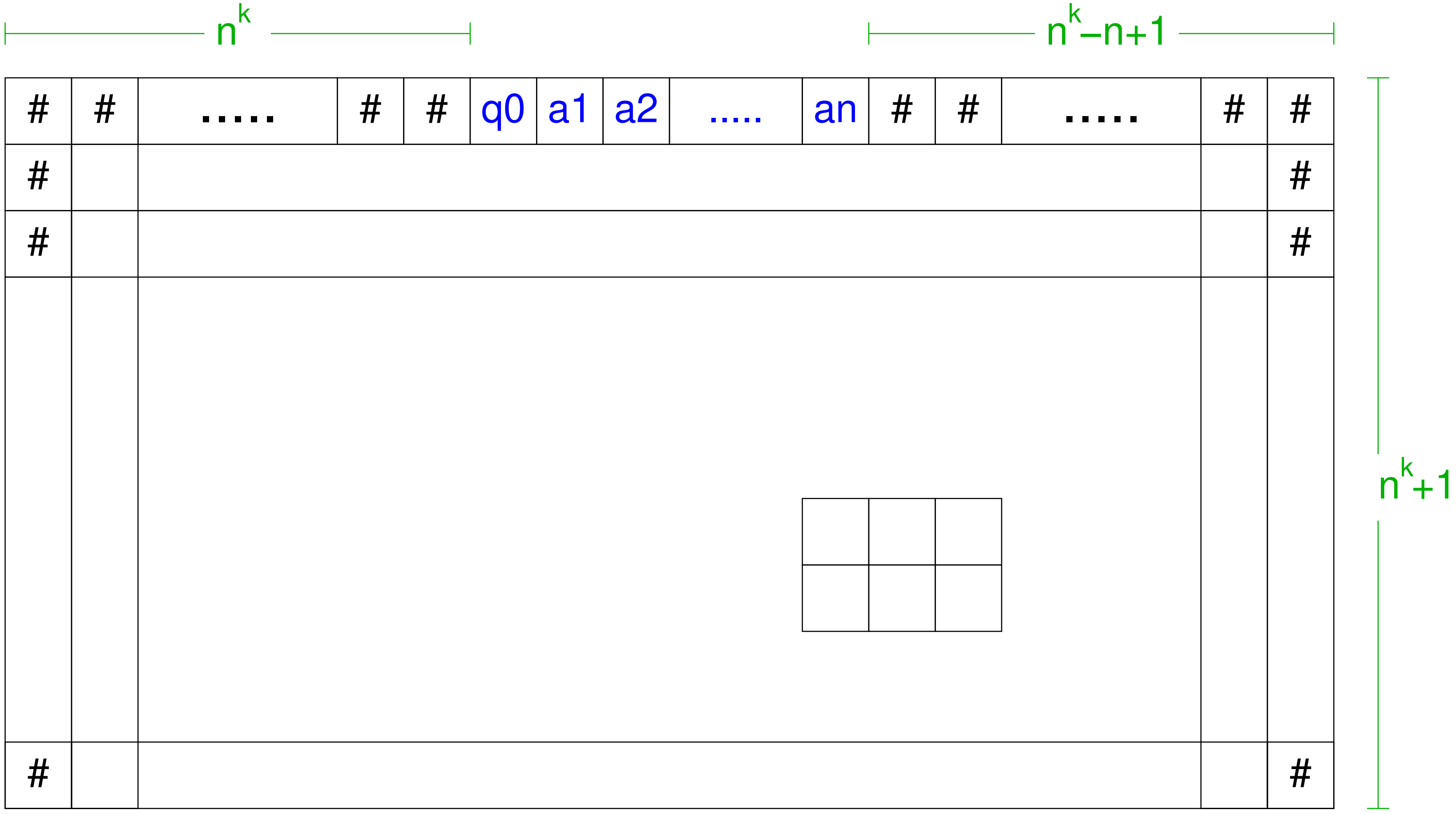
  NTM $N = (Q, \Sigma, \delta, q_0)$ of time complexity n^k

  $w \in \Sigma^*$ induces $\varphi_{N,w}$ such that

  - N accepts $w \iff \varphi_{N,w}$ satifiable
  - length $\varphi_{N,w}$ polynomial in length w

- tableaux

N a NTM deciding the language A in NP in n^k



symbols $S = Q \cup \Sigma$, indices $I = \{\ 0, 1, \ldots, n^k\ \}$, $J = \{\ -n^k, \ldots, n^k + 1\ \}$

variable $x_{i,j,s}$ true iff cell j at row i is symbol s

$$\varphi_{\text{cell}} = \bigwedge_{i \in I, j \in J} \Big[\big(\bigvee_{s \in S} x_{i,j,s} \big) \wedge \big(\bigwedge_{s,t \in S, s \neq t} (\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}}) \big) \Big]$$

in every cell exactly one symbol from S

$$\varphi_{\text{start}} = \big(\bigwedge_{-n^k \le j \le -1} x_{0,j,\#} \big) \wedge x_{0,0,q_0}$$
$$\wedge \big(\bigwedge_{1 \le j \le n} x_{0,j,a_j} \big) \wedge \big(\bigwedge_{n+1 \le j \le n^k+1} x_{0,j,\#} \big)$$

at first line start state q_0, input $w = a_1 a_2 \ldots a_n$, other cells $\#$

$$\varphi_{\text{move}} = \bigwedge_{i \in I', j \in J'} \bigvee_{W(\omega_1, \ldots, \omega_6)} \bigwedge_{0 \le \ell \le 2} (x_{i-1, j+\ell, \omega_\ell} \wedge x_{i, j+\ell, \omega_{3+\ell}})$$

every window with lower-left corner (i, j) is OK

$$\varphi_{\text{accept}} = \bigvee_{i \in I, j \in J} x_{i,j,q_{\text{yes}}}$$

the accepting state is reached

- $W(\omega_1, \ldots, \omega_6)$ the window

ω_1	ω_2	ω_3
ω_4	ω_5	ω_6

is legal

  - $\delta(q, a) \ni (q', b, L)$

c	d	e
c	d	q'

d	e	q
d	q'	e

e	q	a
q'	e	b

q	a	f
e	b	f

a	f	g
b	f	g

  - $\delta(q, a) \ni (q', b, R)$

c	d	q
c	d	b

d	q	a
d	b	q'

q	a	e
b	q'	e

q	e	f
q'	e	f

  - auxiliary

a	b	c
a	b	c

a	b	q_{yes}
a	b	q_{yes}

b	q_{yes}	c
b	q_{yes}	c

q_{yes}	c	d
q_{yes}	c	d

- 3SAT $\{ \varphi \mid \text{3cnf-formula } \varphi \text{ is satisfiable } \}$

- 3SAT is NP-complete
  - 3SAT is in NP
  - inspection proof of Cook's theorem

- translation for w on N

$$\varphi_{N,w} = \varphi_{\text{cell}} \wedge \varphi_{\text{start}} \wedge \varphi_{\text{move}} \wedge \varphi_{\text{accept}}$$

– $\varphi_{\text{cell}} = \bigwedge_{i \in I, j \in J} \left[\left(\bigvee_{s \in S} x_{i,j,s} \right) \wedge \left(\bigwedge_{s,t \in S, s \neq t} (\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}} \right) \right]$

– $\varphi_{\text{start}} = \left(\bigwedge_{-n^k \leq j \leq -1} x_{0,j,\#} \right) \wedge x_{0,0,q_0}$
$\wedge \left(\bigwedge_{1 \leq j \leq n} x_{0,j,a_j} \right) \wedge \left(\bigwedge_{n_1 \leq j \leq n^k+1} x_{0,j,\#} \right)$

– $\varphi_{\text{move}} = \bigwedge_{i \in I', j \in J'} \bigvee_{W(\omega_1,\ldots,\omega_6)} \bigwedge_{0 \leq \ell \leq 2} (x_{i-1,j+\ell,\omega_\ell} \wedge x_{i,j+\ell,\omega_{3+\ell}})$

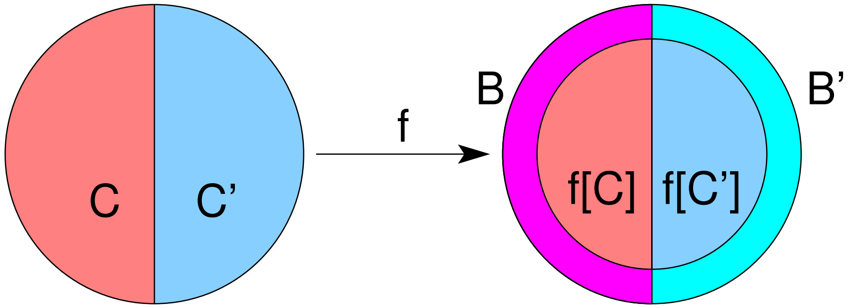
– $\varphi_{\text{accept}} = \bigvee_{i \in I, j \in J} x_{i,j,q_{\text{yes}}}$

can all be turned in to 3cnf format

- reduction lemma

  if B is NP and $C \leq_P B$ with C NP-complete

  then B is NP-complete



f polynomial s.t. $w \in C \iff f(w) \in B$

VERTEX-COVER $= \{ \langle G, k \rangle \mid G$ has k-cover $\}$

- VERTEX-COVER is NP-complete

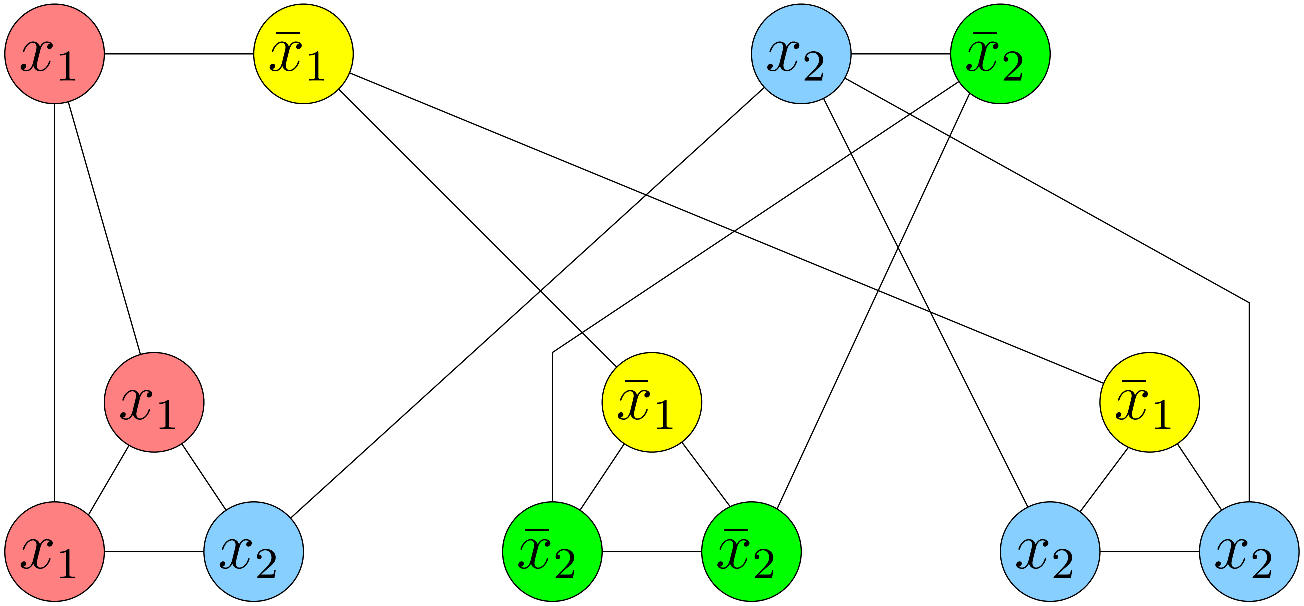
$\varphi = \bigwedge_{1 \le i \le \ell} (a_i \vee b_i \vee c_i)$ with variables $x_1, \ldots, x_m$

nodes $V = \{ u_i, v_i, w_i \mid 1 \le i \le \ell \} \cup \{ s_j, t_j \mid 1 \le j \le m \}$

edges $E = \{ (u_i, v_i), (v_i, w_i), (w_i, u_i) \mid 1 \le i \le \ell \} \cup$
$\{ (s_j, t_j) \mid 1 \le j \le m \} \cup$
$\{ (u_i, s_j) \mid a_i = x_j \} \cup \{ (v_i, s_j) \mid b_i = x_j \} \cup$
$\{ (w_i, s_j) \mid c_i = x_j \} \cup \{ (u_i, t_j) \mid a_i = \bar{x}_j \} \cup$
$\{ (v_i, t_j) \mid b_i = \bar{x}_j \} \cup \{ (w_i, t_j) \mid c_i = \bar{x}_j \}$

graph $G_\varphi = (V, E)$, cover size $k_\varphi = m + 2\ell$

- 3SAT $\leq_P$ VERTEX-COVER



graph for $(x_1 \vee x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee x_2)$

cover size $2 + 3 \cdot 2 = 8$

SUBSETSUM $= \{ \langle X, \mathit{wgt}, t \rangle \mid$ set X has subset of total weight $t \}$

- 3SAT $\leq_P$ SUBSETSUM

$\bigwedge_{1 \leq i \leq k} (a_i \vee b_i \vee c_i)$ with variables $x_1, \ldots, x_\ell$

set $X = \{ y_j, z_j \mid 1 \leq j \leq \ell \} \cup \{ g_i, h_i \mid 1 \leq i \leq k \}$

weight $\mathit{wgt}(y_j) = 10^{k+\ell-j} + \sum_{i=1}^{k} \delta_{i,j} \cdot 10^{k-i}$ where

$$\delta_{i,j} = \begin{cases} 1 & \text{if } x_j = a_i, b_i \text{ or } c_i \\ 0 & \text{otherwise} \end{cases}$$

weight $\mathit{wgt}(g_i) = 10^{k-i}$, weights z_j and h_i similar

target weight $t = \sum_{j=1}^{\ell} 10^{k+\ell-j} + \sum_{i=1}^{k} 3 \cdot 10^{k-i}$

$$(x_1 \vee x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee x_2)$$

	x_1	x_2	c_1	c_2	c_3
y_1	1	0	1	0	0
z_1	1	0	0	1	1
y_2	0	1	1	0	1
z_2	0	1	0	1	0
g_1	0	0	1	0	0
h_1	0	0	1	0	0
g_2	0	0	0	1	0
h_2	0	0	0	1	0
g_3	0	0	0	0	1
h_3	0	0	0	0	1
	1	1	3	3	3

wgt (y_1)	$=$	10100
wgt (z_1)	$=$	10011
wgt (y_2)	$=$	1101
wgt (z_2)	$=$	1010
wgt (g_1)	$=$	100
wgt (h_1)	$=$	100
wgt (g_2)	$=$	10
wgt (h_2)	$=$	10
wgt (g_3)	$=$	1
wgt (h_3)	$=$	1
t	$=$	11333

$\neq$-SAT $= \{ \phi \in$ 3CNF $\mid \phi$ has a $\neq$-assignment $\}$

in each clause both a true and a false literal

- 3SAT $\leq_P$ $\neq$-SAT

replace $a_i \vee b_i \vee c_i$ by $(a_i \vee b_i \vee y_i) \wedge (\bar{y}_i \vee c_i \vee z)$

where y_i and z are fresh and z shared among all clauses

$(x_1 \vee x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee x_2)$

translates into

$(x_1 \vee x_1 \vee y_1) \wedge (\bar{y}_1 \vee x_2 \vee z) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee y_2) \wedge (\bar{y}_2 \vee \bar{x}_2 \vee z) \wedge (\bar{x}_1 \vee x_2 \vee y_3) \wedge (\bar{y}_3 \vee x_2 \vee z)$

if α is a $\neq$-assigment, then so is its conjugate $\bar{\alpha}$

3COLORING = $\{ \langle G \rangle \mid G$ can be colored with 3 colors $\}$

- $\neq$-SAT $\leq_P$ 3COLORING

$\varphi = \bigwedge_{1 \leq i \leq k} (a_i \vee b_i \vee c_i)$ with variables $x_1, \ldots, x_\ell$

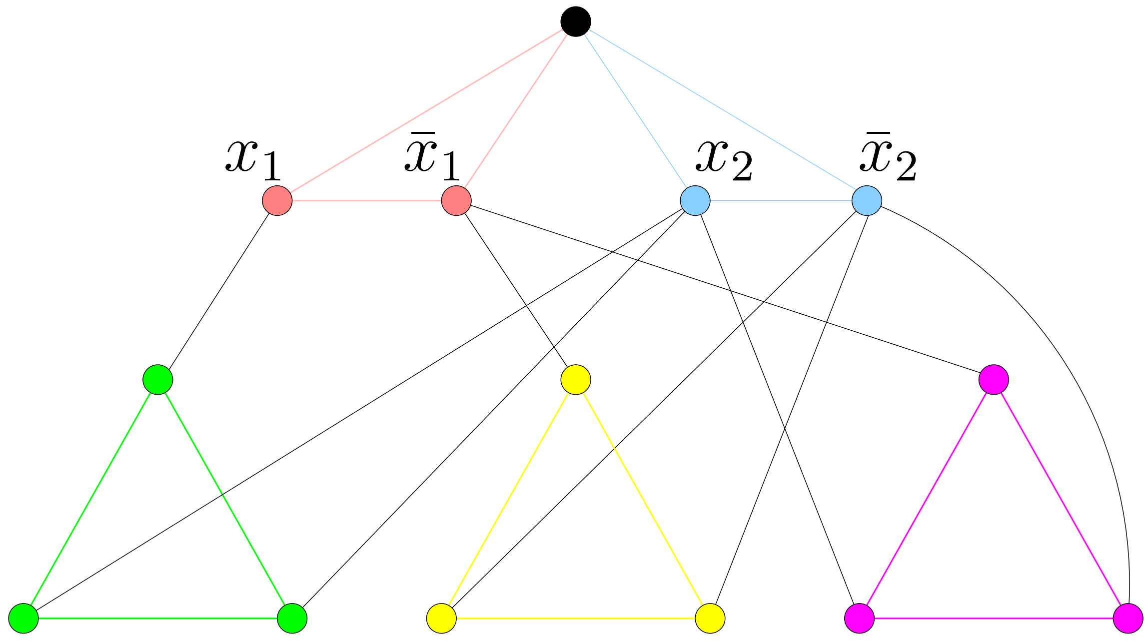
nodes $\{ u_i, v_i, w_i \mid 1 \leq i \leq k \} \cup \{ x_j, \bar{x}_j \mid 1 \leq j \leq \ell \} \cup \{ z \}$

edges $\{ (u_i, v_i), (v_i, w_i), (w_i, u_i) \mid 1 \leq i \leq k \} \cup$
$\{ (z, x_j), (x_j, \bar{x}_j), (\bar{x}_j, z) \mid 1 \leq j \leq \ell \} \cup$
$\{ (u_i, x_j) \mid u_i = x_j \} \cup \{ (u_i, \bar{x}_j) \mid u_i = \bar{x}_j \} \cup \ldots$

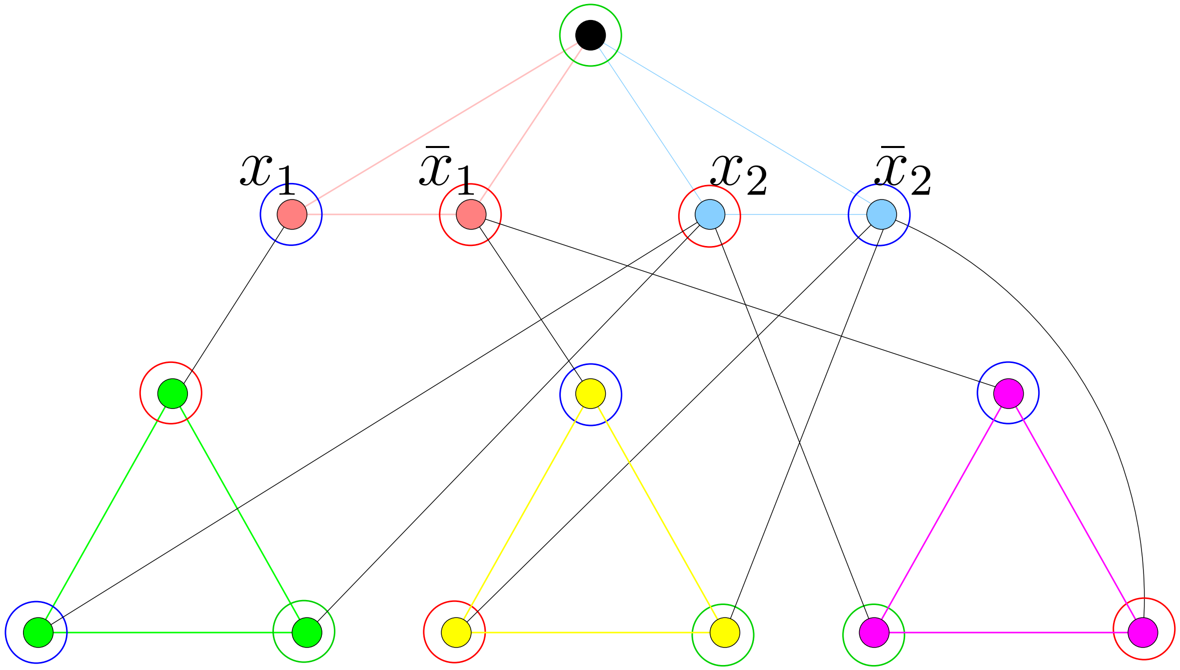
$\alpha(x_j)$	x_j	$\bar{x}_j$	z
T	R	B	G
F	B	R	G

$\alpha(a_i)$	$\alpha(b_i)$	$\alpha(c_i)$	$\gamma(u_i)$	$\gamma(v_i)$	$\gamma(w_i)$
T	F	–	B	R	G
T	T	F	B	G	R
F	T	–	R	B	G
F	F	T	R	G	B

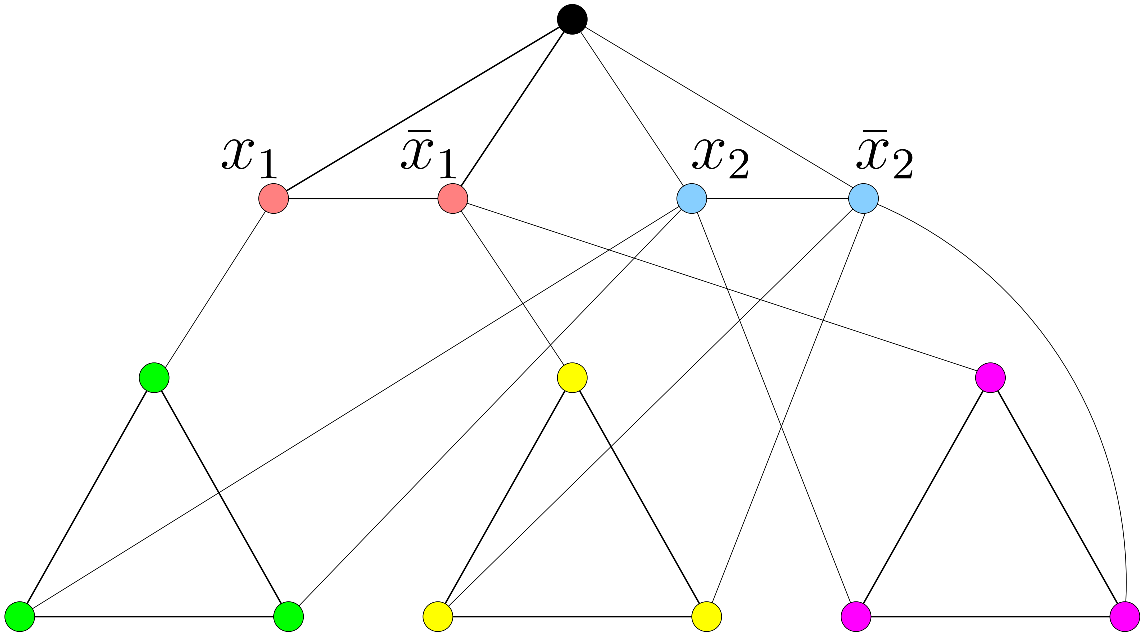
$$(x_1 \vee x_2 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_2)$$



$$(x_1 \vee x_2 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_2)$$



$$(x_1 \vee x_2 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_2)$$



$$(x_1 \vee x_2 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_2)$$