On a functor for probabilistic bisimulation and preservation of weak pullbacks

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Abstract The preservation of weak pullbacks is studied for a functor $\mathcal{M}_1$ on the category **UMS** of ultrametric spaces and nonexpansive mappings. The functor $\mathcal{M}_1$ associates with an ultrametric space its collection of Borel probability measures with compact support. By application of the Max-flow Min-cut Theorem of graph theory a mediating morphism for a weak pullback diagram can be constructed in **SET**. It is shown that the functor $\mathcal{M}_1$ as functor from **UMS** to **SET** preserves weak pullbacks. A possible solution for the preservation of weak pullbacks by $\mathcal{M}_1$ as **UMS**-functor is indicated.

Keywords Ultrametric spaces, Borel measures, coalgebras, preservation of weak pullbacks

1 Introduction

The notion of a coalgebra turns out to be one of the key concepts for an understanding of bisimulation from a category theoretical perspective. See, e.g., [AM89, RT93, Rut96, JR97]. In the coalgebraic approach bisimulation can be considered with respect to any functor. Of course, some functors are more appealing to use for the description of computational phenomena than others. Final coalgebra theorems provide conditions for a functor that guarantee the existence of 'greatest' solution of domain equations, usually by exploiting the structure of the underlying category. See [RT93, Bar93, Mon98]. Another useful feature concerns the preservation of weak pullbacks by a functor. This property does not only imply transitivity of the notion of F-bisimulation induced by a functor F, but is also fundamental for the construction of a final semantics. An important consequence of the property of preserving weak pullbacks is that F-bisimulation in any F-coalgebra coincides with equality in the final coalgebra of F (cf. [Rut96, Theorem 9.3]).

For discrete probabilistic transition systems the notion of Larsen-Skou bisimulation as introduced in [LS91] is the standard process equivalence. This probabilistic bisimulation can be captured categorically by a **SET**-functor, which, moreover, weakly preserves pullbacks. See [Mos97, VR97a]. In [VR97a], additionally a generalization $\mathcal{M}_1$ of this functor is proposed, with Borel probability measure replacing the discrete distributions. Is in proven in the technical report [VR97b], that $\mathcal{M}_1$ is well-defined, preserves completeness and is locally nonexpansive. Thus the functor can be employed for the construction of semantical domains via metric domain equations á là America/Rutten (cf. [AR89, RT93]). Also, it is shown that a suitable functor based on $\mathcal{M}_1$ possesses a final coalgebra. However, for $\mathcal{M}_1$ to be a first-class citizen in the world of categorical bisimulation it should preserve weak pullbacks. In this note only a partial result can be obtained for this. We are able to show that $\mathcal{M}_1$ preserves weak pullbacks as a functor from **UMS**, the category of ultrametric spaces and nonexpansive mappings, to **SET**. In particular, it is not known to us how to assure that the mediating morphism of a pullback diagram can be shown to be nonexpansive. In case such a function can be proven nonexpansive we immediately obtain a corresponding preservation result for $\mathcal{M}_1$ as functor from **UMS** to **UMS**.

Related work, for the research report here, is the paper [BDEP97] a notion of bisimulation for what are called labeled Markov processes is given. Their process equivalence is based on the idea of spans of zig-zags (cf. [JNW96, WN97]). Blute et al. combine Polish spaces and conditional probability distributions, also called stochastic kernels. Their results do not fit in neatly in the coalgebraic approach, but, more importantly, they do have that pullbacks exists in the category they are working with and, as a consequence, the notion of bisimulation based on spans of zig-zags is an equivalence relation. The precise relationship between the approach reported in [BDEP97] and that of [VR97a] and the implications of the follow-up paper [DEP97] are topics of further investigations.

The remainder of this paper is organized as follows. In Section 2 the various prerequisites of metric topology, category theory and graph theory are briefly discussed. In Section 3 the notion of a Borel measure with compact support is given. This section also gathers the basic properties that are exploited in Section 4. In this section the functor $\mathcal{M}_1$ on the category **UMS** is presented. The main body of the section constitutes the contribution of the paper. It is devoted to the technical ingredients underlying our partial preservation result. In particular, the max-flow min-cut theorem is applied for the construction of a suitable probability measure. The last section, Section 5, deals with the preservation of weak pullbacks. A theorem for the preservation of weak pullbacks by $\mathcal{M}_1$ as functor to **SET** is given and a possible modification in order to obtain a result for $\mathcal{M}_1$ as **UMS**-functor is indicated.

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2 Mathematical preliminaries

Metric topology We assume familiarity with the elementary concepts of metric topology such as metric space, continuous function, compact subset and completeness (cf. [Dug66, Eng89, BV96, Smy92]). A metric space (M, d) is called an ultrametric space if d satisfies the strong triangle inequality, i.e.

$$\forall x, y, z \in M \colon d(x, z) \leq \max\{\, d(x, y), d(y, z) \,\}.$$

For $\varepsilon > 0$, $B_\varepsilon(x)$ is the open ball around x of radius ε in M. We use $\mathcal{O}(M)$ to denote the collection of open sets of M with respect to a metric d. So $\mathcal{O}(M)$ is collection of all arbitrary unions of open balls in (M, d). We furthermore write $\mathcal{O}_n(M)$ for the collection of arbitrary unions of balls of radius 2^{-n}, and $\mathcal{O}_*(M)$ for the union of all the $\mathcal{O}_n(M)$, $n \in \mathbb{N}$. Also, $\mathcal{O}_\varepsilon(M)$ is the collection of all open balls of radius ε in M. Below we will employ the notation $X + \varepsilon$, for a subset $X \subseteq M$ and $\varepsilon > 0$, where

$$X + \varepsilon = \{\, B_\varepsilon \mid x \in M \,\}.$$

If (M, d) is a metric space, we then can endow the powerset $\mathcal{P}(M)$ with the so-called Hausdorff-distance d_H given by

$$d_H(X, Y) = \inf\{\, \varepsilon > 0 \mid X \subseteq Y + \varepsilon,\ Y \subseteq X + \varepsilon \,\}.$$

The Hausdorff-distance is generally a pseudo-metric; the collection $\mathcal{P}_{co}(M)$ of compact subsets of M supplied with the Hausdorff-distance is a metric space. The space $(\mathcal{P}_{co}(M), d_H)$ is an ultrametric space if d is an ultrametric on M. It is complete, if (M, d) is complete. (See,e,g, [BV96] for more detail.) In an ultrametric space we have that $X + \varepsilon = Y + \varepsilon$ iff $(X + \varepsilon) \cap (Y + \varepsilon) \neq \emptyset$.

Category theory (see, e.g., [BW90, Rut96]) Let $\mathbf{A}$ be a category, and B, C, D and $f: B \to D$, $g: C \to D$ be objects and arrows in $\mathbf{A}$. A triple (A, π_1, π_2) consisting of an object A and two arrows $\pi_1: A \to B$, $\pi_2: A \to C$ in $\mathbf{A}$ is called a pullback for f and g if the following two conditions are fulfilled:

- $f \circ \pi_1 = g \circ \pi_2$.
- If the triple (X, ϕ, ψ) with X an object and $\phi: X \to B$, $\psi: X \to C$ arrows such that $f \circ \phi = g \circ \psi$, then there exists a unique arrow $\chi: X \to A$ such that $\phi = \pi_1 \circ \chi$ and $\psi = \pi_2 \circ \chi$. The arrow χ is referred to as the mediating morphism.

The triple (A, π_1, π_2) is called a weak pullback for f and g if the two conditions are satisfied except, possibly, for the uniqueness of the mediating arrows.

A category $\mathbf{A}$ is said to have pullbacks, if a pullback exists for any two arrows in $\mathbf{A}$. For example, the category $\mathbf{UMS}$ of ultrametric spaces and nonexpansive mappings has pullbacks: Suppose $f: B \to D$ and $g: C \to D$. Put $A = \{ (b, c) \mid b \in B, c \in C, f(b) = g(c) \}$. Consider A as a subspace of the ultrametric space $B \times C$. Then (A, π_1, π_2) —with $\pi_1: A \to B$, $\pi_2: A \to C$ the projections on the first and on the second component— is a pullback for f and g. Clearly $f \circ \pi_1 = g \circ \pi_2$ by definition of A. The mappings π_1, π_2 are nonexpansive. Moreover, if (X, ϕ, ψ) satisfies $f \circ \phi = g \circ \psi$, a mediating morphism is uniquely determined by the equation $\chi(x) = (\phi(x), \psi(x))$, for all $x \in X$. Such an arrow is nonexpansive. (Note that $A \subseteq B \times C$ is a closed. If $(b, c) \notin A$, then $f(b) \neq g(c)$ and thus they have a positive distance ε, say. It follows that the open set $B_\varepsilon(b) \times B_\varepsilon(c)$ is an open neighborhood of (b, c) which misses A.)

A functor $F: \mathbf{A} \to \mathbf{B}$ from a category $\mathbf{A}$ to a category $\mathbf{B}$ is said to transform pullbacks into weak pullbacks if, for any two arrows f and g in and any pullback (A, π_1, π_2) of f and g it holds that $(F(A), F(\pi_1), F(\pi_2))$ is a weak pullback of $F(f)$ and $F(g)$ in $\mathbf{B}$.

Consider, for example, the functor $\mathcal{P}_{co}$ on $\mathbf{UMS}$ which maps an ultrametric space M to its collection $\mathcal{P}_{co}(M)$ of compacta endowed with the Hausdorff-distance and maps a nonexpansive function $f: M \to N$ to the nonexpansive direct-image function $\mathcal{P}_{co}(f): \mathcal{P}_{co}(M) \to \mathcal{P}_{co}(N)$ given by $\mathcal{P}_{co}(f)(K) = f[K]$ for any compact $K \subset M$. Suppose (A, π_1, π_2) is the pullback of two morphisms f and g as constructed above. By functoriallity we have $\mathcal{P}_{co}(f) \circ \mathcal{P}_{co}(\pi_1) = \mathcal{P}_{co}(g) \circ \mathcal{P}_{co}(\pi_2)$. Moreover, if $K \subseteq B$, $L \subseteq C$ are compact then $H = (K \times L) \cap A$ is the intersection of a compact and a closed set in $B \times C$, hence H is compact. One can straightforwardly check that $\pi_1[H] = K$ and $\pi_2[H] = L$ provided $f[K] = g[L]$. Moreover, by definition of the distance on $B \times C$, we have $d(H_1, H_2) \leq \max\{ d(K_1, K_2), d(L_1, L_2) \}$ with $K_1, K_2 \in \mathcal{P}_{co}(B)$, $L_1, L_2 \in \mathcal{P}_{co}(C)$ and H_1, H_2 as expected. Now suppose (X, ϕ, ψ) satisfies $\mathcal{P}_{co}(f) \circ \phi$ and $\mathcal{P}_{co}(g) \circ \psi$. It follows from the above that the assignment $x \mapsto (\phi(x), \psi(x)) \cap A$ defines a nonexpansive mapping $\chi: X \to \mathcal{P}_{co}(A)$ such that $\phi = \pi_1 \circ \chi$ and $\psi = \pi_2 \circ \chi$. We conclude that $(\mathcal{P}_{co}(A), \mathcal{P}_{co}(\pi_1), \mathcal{P}_{co}(\pi_2))$ is a weak pullback for $\mathcal{P}_{co}(f)$ and $\mathcal{P}_{co}(g)$, and that $\mathcal{P}_{co}$ on $\mathbf{UMS}$ transforms pullbacks into weak pullbacks.

The functor F is said to preserve weak pullbacks if $(F(A), F(\pi_1), F(\pi_2))$ is a weak pullback of $F(f)$ and $F(g)$ in $\mathbf{B}$ in case (A, π_1, π_2) is a pullback of f and g in $\mathbf{A}$. In [Mos97, VR97b] it is shown that **SET**-functor delivering discrete probability distributions preserves weak pullbacks. The proof in [VR97b] employs a max-flow min-cut theorem for this (see below). By diagram chasing one can easily show: if a category $\mathbf{A}$ has pullbacks and a functor $F: \mathbf{A} \to \mathbf{B}$ transforms pullbacks into weak pullbacks, then F preserves weak pullbacks.

Graph theory (see, e.g., [Bol79]) A network N is a 4-tuple $N = \langle G, c, s, t \rangle$ where $G = \langle V, E \rangle$ is a directed graph with set of vertices V and set of edges E, $c: E \to [0,1]$ a mapping called the capacity function, and $s, t \in V$ two distinct vertices.

A flow f for N from s to t is a function $f: E \to [0,1]$ such that

– $\forall v \in V \setminus \{s,t\}$: $\sum\{ f(u,v) \mid u \in V, (u,v) \in E \} = \sum\{ f(v,u) \mid u \in V, (v,u) \in E \}$

– $\forall e \in E$: $f(e) \leq c(e)$.

The value of a flow f is given by

$$(\sum\{ f(s,v) \mid v \in V \setminus \{s\}, (s,v) \in E \}) - (\sum\{ f(v,s) \mid v \in V \setminus \{s\}, (v,s) \in E \}).$$

A flow is called maximal if its value is maximal over all flows for the network.

A cut C for N is a subset $C \subseteq V$ such that $s \in C$ and $t \notin C$. The value of the cut is given by $\sum\{ c(v,v') \mid v \in C, v' \notin C, (v,v') \in E \}$. A cut is called minimal if its value is minimal over all cuts for the network.

Theorem 2.1 *(Max-flow Min-cut Theorem) Any network has a maximal flow and a minimal cut. Their values are equal.*

Proof See [Bol79, p47]. □

3 Borel probability measures with compact support

First, we briefly recall some basic measure-theoretic definitions. See, e.g., the standard texts [Hal74, Rud66] for more details. A collection $\mathcal{A}$ of subsets of a set X is called a σ-algebra if X is in $\mathcal{A}$ and $\mathcal{A}$ is closed under complement and countable unions. It follows that if $E_1, E_2 \in \mathcal{A}$ then also $E_1 \cap E_2, E_1 \setminus E_2 \in \mathcal{A}$. For an ultrametric space M, the smallest σ-algebra containing all open sets, i.e. the intersection of all σ-algebras $\mathcal{A}$ with $\mathcal{O}(M) \subseteq \mathcal{A}$, is called the Borel σ-algebra; notation $Bor(M)$. An element $E \in Bor(M)$ is called a Borel-set of M.

A function $\mu: Bor(M) \to [0,1]$ is called a Borel probability measure on M if $\mu(M) = 1$ and $\mu(\bigcup_{i \in I} E_i) = \sum_{i \in I} \mu(E_i)$ for any countable collection $\{ E_i \mid i \in I \}$ in $Bor(M)$ of pairwise disjoint Borel-sets. The latter property is referred to as σ-additivity. If the property holds for finite unions we say that μ is finitely additive. As $\mu(\emptyset) = 0$, σ-additivity implies finite additivity. A function $\nu: \mathcal{O}_*(M) \to [0,1]$ is called σ-additive if $\nu(\bigcup_{i \in I} O_i) = \sum_{i \in I} \nu(O_i)$ for any countable collection $\{ O_i \mid i \in I \}$ in $\mathcal{O}_*(M)$ such that $\bigcup_{i \in I} O_i \in \mathcal{O}_*(M)$.

From the finite additivity it follows that a Borel probability measure is monotone, i.e., $\mu(E_1) \leq \mu(E_2)$ if $E_1 \subseteq E_2$. From the definition we obtain for a Borel probability measure μ that

$$\mu(\emptyset) = 0 \text{ and } \mu(E_1) = \mu(E_1 \cap E_2) + \mu(E_1 \setminus E_2).$$

Borel probability measures are determined by their value on opens. In particular we have the following theorem.

Theorem 3.1 *If M is an ultrametric space and μ is a Borel probability measure on M. Then μ is regular, i.e.,*

$$\mu(E) = \inf\{ \mu(O) \mid O \in \mathcal{O}(M), E \subseteq O \}$$

for any Borel-set $E \subseteq M$. Moreover, each σ-additive function $\nu: \mathcal{O}_*(M) \to [0,1]$ such that $\nu(M) = 1$ extends uniquely to a Borel probability measure on M.

Proof See [Hal74, §11–13]. □

Intuitively the support of measure is the set that carries the mass of the measure; outside the support the measure vanishes (cf. [Bar88]). In the next section we will discuss a functor that yields for an ultrametric space the collection of all Borel probability measures of compact support. The restriction to measure of compact support is essential for the completeness result as obtained in [VR97b].

Definition 3.2 Let M be an ultrametric space and let μ be a Borel probability measure on M. The support $spt(\mu)$ of μ is given by

$$spt(\mu) = \{ x \in M \mid \forall O \in \mathcal{O}(M): x \in O \Rightarrow \mu(O) \neq 0 \}.$$

The measure μ is said to have a compact support if $spt(\mu)$ is a compact subset of M.

The next elementary property will be used in the proof of Lemma 3.4.

Lemma 3.3 *Let μ be a Borel probability measure on an ultrametric space M. If $E_1, E_2 \subseteq M$ are Borel-sets such that $\mu(E_1), \mu(E_2) = 1$, then $\mu(E_1 \cap E_2) = 1$.*

Proof Straightforward. We have $\mu(E_1), \mu(E_2), \mu(E_1 \cup E_2) = 1$. Thus

$$\begin{aligned} \mu(E_1 \cup E_2) &= \mu(E_1 \setminus E_2) + \mu(E_1 \cap E_2) + \mu(E_2 \setminus E_1) \\ \mu(E_1) &= \mu(E_1 \setminus E_2) + \mu(E_1 \cap E_2) \\ \mu(E_2) &= \mu(E_1 \cap E_2) + \mu(E_2 \setminus E_1) \end{aligned}$$

from which it follows that $\mu(E_1 \setminus E_2), \mu(E_2 \setminus E_1) = 0$ and $\mu(E_1 \cap E_2) = 1$. □

The following result gives a necessary and sufficient condition for a Borel probability measure to have a compact support. There should be a compact subset of measure 1.

Lemma 3.4 *Let M be an ultrametric space and μ a Borel probability measure on M. Suppose $\tilde{K} \subseteq M$ is compact such that $\mu(\tilde{K}) = 1$. Then μ has a compact support K_0 and $\mu(K_0) = 1$.*

Proof Define the collection $\mathcal{K} = \{ K \mid K \subseteq \tilde{K}$ compact, $\mu(K) = 1 \}$. Put $K_0 = \bigcap \mathcal{K}$. Then we have that K_0 is compact as closed subset of $\tilde{K}$.

Let $O \in \mathcal{O}(M)$ be such that $K_0 \subseteq O$. Suppose that for no $K \in \mathcal{K}$ it holds that $K \subseteq O$. Consider the collection $\mathcal{K}' = \{ K \setminus O \mid K \in \mathcal{K} \}$. Let $K'_1, \ldots, K'_n \in \mathcal{K}'$. Say $K'_i = K_i \setminus O$ with $K_i \in \mathcal{K}$. Put $K = \bigcap_{i=1}^n K_i$. We then have that K is a compact subset of $\tilde{K}$ and, by Lemma 3.3, that $\mu(K) = 1$. Hence $K \in \mathcal{K}$. By assumption $K \not\subseteq O$. Therefore $K \setminus O = \bigcap_{i=1}^n K'_i \neq \emptyset$. It follows that the collection $\mathcal{K}'$ has the finite intersection property. Hence, by compactness of $\tilde{K}$, $\bigcap \mathcal{K}' \neq \emptyset$. However, for $x \in \bigcap \mathcal{K}'$ we have both $x \in O$ and $x \in \bigcap \mathcal{K} = K_0 \subseteq O$. Contradiction. We conclude $K \subseteq O$ for some $K \in \mathcal{K}$. From this we obtain $\mu(O) = \mu(K) = 1$. By regularity of μ (cf. Theorem 3.1) it follows that $\mu(K_0) = 1$.

We check that $K_0 = spt(\mu)$. Suppose, on the one hand, $x \in K_0$. If $\varepsilon > 0$, then $K_0 \setminus B_\varepsilon(x)$ is a strict compact subset of K_0, hence $\mu(K_0 \setminus B_\varepsilon(x)) < 1$, thus $\mu(B_\varepsilon(x)) \neq 0$. It follows that $x \in spt(\mu)$. Suppose, on the other hand, $x \notin K_0$. Pick, using the compactness of K_0, $\varepsilon > 0$ be such that $B_\varepsilon(x) \cap K_0 = \emptyset$. Clearly $\mu(B_\varepsilon(x)) = 0$, thus $x \notin spt(\mu)$. □

Suppose M and N are ultrametric spaces and $f: M \to N$ is continuous. A basic fact from measure theory is that, for a Borel probability measure μ on M, the so-called image measure $\mu \circ f^{-1}$ given by

$$(\mu \circ f^{-1})(E) = \mu(f^{-1}(E))$$

for $E \in Bor(N)$, is a well-defined Borel probability measure on N. The next lemma states that if μ has a compact support K, the image measure $\mu \circ f^{-1}$ has also a compact support, viz. $f[K]$.

Lemma 3.5 *Let M, N be two ultrametric spaces. If μ is a Borel probability measure on M with a compact support and $f: M \to N$ is nonexpansive, then the function $\nu: \mathrm{Bor}(N) \to [0,1]$ given by $\nu(E) = \mu(f^{-1}(E))$, for $E \subseteq N$ Borel, is a Borel probability measure with compact support. In particular, $spt(\nu) = f[spt(\mu)]$.*

Proof By the properties of f^{-1} it follows that ν is σ-additive. If $(E_i)_i$ is a disjoint sequence of Borel sets in N, then $(f^{-1}(E_i))_i$ is a disjoint sequence of Borel sets in M. hence

$$\nu(\bigcup_i E_i) = \mu(\bigcup_i f^{-1}(E_i)) = \mu(f^{-1}(\bigcup_i E_i)) = \nu(\bigcup_i E_i)$$

by σ-additivity of μ. Moreover $\nu(N) = \mu(M) = 1$.

Let $K = spt(\mu)$ and put $L = f[K]$. Then L is the continuous image of a compactum, hence compact in N. If $y \in L$, then $y = f(x)$ for some $x \in K$. Thus, for all $\varepsilon > 0$, $\nu(B_\varepsilon(y)) = \mu(f^{-1}(B_\varepsilon(y))) \geq \mu(B_\delta(x)) > 0$ for a suitable δ such that $f(B_\delta(x)) \subseteq B_\varepsilon(y)$, as x is in the support of μ. Hence $y \in spt(\nu)$. If $y \notin L$, then we can find $\varepsilon > 0$ such that $B_\varepsilon(y) \cap L = \emptyset$. Thus $f^{-1}(B_\varepsilon(y)) \cap f^{-1}(L) = \emptyset$. In particular, $f^{-1}(B_\varepsilon(y)) \cap K = \emptyset$, from which we derive $\nu(B_\varepsilon(y)) = \mu(f^{-1}(B_\varepsilon(y)) = 0$ and $y \notin spt(\nu)$. Hence, $L = spt(\nu)$ and $spt(\nu) = f[spt(\mu)]$ follows. □

4 An application of the Max-flow Min-cut Theorem

The category **UMS** has ultrametric spaces as its objects and nonexpansive mappings as its arrows. The functor $\mathcal{M}_1$ is defined on **UMS**. In this section we present, given suitable measures β and γ, a construction of a measure α such that $\mathcal{M}_1(\pi)(\alpha) = \beta$ and $\mathcal{M}_1(\pi_2) = \gamma$. The construction will be used in Section 5 for a preservation result of $\mathcal{M}_1$.

Definition 4.1 The functor $\mathcal{M}_1$: **UMS** $\to$ **UMS** is given as follows

- $\mathcal{M}_1(M, d) = (\{\, \mu \mid \mu \text{ a Borel probability measure of compact support} \,\}, d_m)$
- $\mathcal{M}_1(f)(E) = \mu(f^{-1}(E))$ for $M \in$ **UMS**, $f: M \to N \in$ **UMS**, $\mu \in \mathcal{M}_1(M)$, $E \subseteq N$ Borel

where the distance d_m on $\mathcal{M}_1(M,d)$ is given by

$$d_m(\mu,\nu) = \inf\{\, \varepsilon > 0 \mid \forall O \in \mathcal{O}_\varepsilon(M)\colon \mu(O) = \nu(O) \,\}$$

for Borel probability measures μ, ν on M with a compact support. We will write $\mathcal{M}_1(M)$ instead of $\mathcal{M}_1(M,d)$ when convenient.

In [VR97a, VR97b] it is discussed that the distance d_m is indeed an ultrametric and that the functor $\mathcal{M}_1$ is well-defined, preserves completeness and is locally nonexpansive on **UMS**. The remainder of this section is devoted to the result (cf. Theorem 4.9) underpinning the preservation of weak pullbacks by $\mathcal{M}_1$ from **UMS** to **SET** as treated in Section 5.

We fix two morphisms $f\colon \mathcal{B} \to \mathcal{D}$, $g\colon \mathcal{C} \to \mathcal{D}$ in **UMS** and two Borel probability measures $\beta \in \mathcal{M}_1(\mathcal{B})$, $\gamma \in \mathcal{M}_1(\mathcal{C})$ such that $(\mathcal{M}_1(f))(\beta) = (\mathcal{M}_1(g))(\gamma)$. We will construct a Borel probability measure α on $\mathcal{A}$, where $(\mathcal{A}, \pi_1, \pi_2)$ is the pullback of f and g in **UMS**, such that $(\mathcal{M}_1(\pi_1))(\alpha) = \beta$ and $(\mathcal{M}_1(\pi_2))(\alpha) = \gamma$. In this section $\mathcal{O}_n$, $\mathcal{O}_*$ and $\mathcal{O}$ are short for $\mathcal{O}_n(B \times C)$, $\mathcal{O}_*(B \times C)$ and $\mathcal{O}(B \times C)$, respectively.

Let the compact subset $K \subseteq \mathcal{B}$ be the support of β and the compact subset $L \subseteq \mathcal{C}$ be the support of γ. For $n \in \mathbb{N}$, $\mathcal{B}_n$ and $\mathcal{C}_n$ denote the finite collections of open balls in $\mathcal{B}$ and $\mathcal{C}$ of radius 2^{-n} intersecting K and L, respectively. Let $\kappa(n)$ and $\lambda(n)$ denote the finite cardinality of $\mathcal{B}_n$ and $\mathcal{C}_n$. From Lemma 3.5 it follows that $f[K] = spt((\mathcal{M}_1(g))(\beta))$ and $g[L] = spt((\mathcal{M}_1(g))(\gamma))$. Hence, by the assumption $(\mathcal{M}_1(f))(\beta) = (\mathcal{M}_1(g))(\gamma)$, $f[K] = g[L]$.

Next we will employ the Max-flow Min-cut Theorem for the possible values for the Borel probability measure α, that is sought for, on open 'squares' of size 2^{-n}. The idea of using the network result is taken from [Jon89].

Lemma 4.2 *Let $n \in \mathbb{N}$. Fix two distinct and fresh symbols src and snk. Define the network Net_n as follows:*

- *Net_n has the set $\{src\} \cup \mathcal{B}_n \cup \mathcal{C}_n \cup \{snk\}$ as its set of nodes*
- *for every $B \in \mathcal{B}_n$ there is an edge (src, B) of capacity $\beta(B)$*
- *for every $C \in \mathcal{C}_n$ there is an edge (C, snk) of capacity $\gamma(C)$*
- *there is an edge (B,C), for $B \in \mathcal{B}_n$ and $C \in \mathcal{C}_n$ iff $f[B]+2^{-n} = g[C]+2^{-n}$; edges (B,C) have capacity 1.*

Then the network Net_n has minimal cut value 1.

Proof If, for $B \in \mathcal{B}_n$ and $C \in \mathcal{C}_n$, it holds that $f[B] \cap g[C] \neq \emptyset$, then, by ultrametricity, the edge (B,C) is in Net_n. Pick $B \in \mathcal{B}_n$. Then, by definition of $\mathcal{B}_n$, $B \cap K \neq \emptyset$, hence $f[B] \cap f[K] \neq \emptyset$. Since $f[K] = g[L]$, we clearly have $f[B] \cap g[L] \neq \emptyset$. By definition of $\mathcal{C}_n$ we have $L \subseteq \bigcup\{\, C \mid C \in \mathcal{C}_n \,\}$, thus $g[L] \subseteq \bigcup\{\, g[C] \mid C \in \mathcal{C}_n \,\}$. It follows that $f[B] \cap g[C] \neq \emptyset$ for some $C \in \mathcal{C}_n$. Therefore, for all $B \in \mathcal{B}_n$ there exists $C \in \mathcal{C}_n$ such that the edge (B,C) is in Net_n, and, vice versa, for all $C \in \mathcal{C}_n$ there exists $B \in \mathcal{B}_n$ such that the edge (B,C) is in Net_n.

Suppose Γ is a cut for Net_n, hence $src \in \Gamma$, $snk \notin \Gamma$. We check that Γ has at least the value 1. Without loss of generality there is no edge (B,C) in the cut, i.e. no edge (B,C) such that $B \in \Gamma \wedge C \notin \Gamma$ or $B \notin \Gamma \wedge C \in \Gamma$, since such edges have capacity 1. Define the open subset $D \subset \mathcal{D}$ by

$$D = (\bigcup\{ g[C] \mid C \in \mathcal{C}_n \cap \Gamma \}) + 2^{-n}.$$

Claim: $\forall B \in \mathcal{B}_n \setminus \Gamma$: $(f[B] + 2^{-n}) \cap D = \emptyset$ and $\forall C \in \mathcal{C}_n \setminus \Gamma$: $(g[C] + 2^{-n}) \cap D = \emptyset$. *Proof of the claim*: If $(f[B] + 2^{-n}) \cap D \neq \emptyset$, then, for some $C \in \mathcal{C}_n \cap \Gamma$, $f[B] + 2^{-n} = g[C] + 2^{-n}$, for $d \in D$ is such that $d(d, f(b)) < 2^{-n}$ for $b \in B$ with $B \in \mathcal{B}_n \setminus \Gamma$ and $d(d, g(c)) < 2^{-n}$ for $c \in C$ with $C \in \mathcal{C}_n \cap \Gamma$, then $d(f(b), g(c)) < 2^{-n}$ and $f[B] + 2^{-n} = g[C] + 2^{-n}$ follows. Since there is no edge (B, C) with $B \notin \Gamma$ and $C \in \Gamma$ it follows that $(f[B] + 2^{-n}) \cap D = \emptyset$ for $B \in \mathcal{B}_n \setminus \Gamma$. Consequently, for $C \in \mathcal{C}_n \setminus \Gamma$ there exists $B \in \mathcal{B}_n$ such that the edge (B, C) is in Net_n. We have $f[B] + 2^{-n} = g[C] + 2^{-n}$ and, by the assumption on edges (B, C), $B \in \mathcal{B}_n \setminus \Gamma$. It follows, from the first property that $(g[C] + 2^{-n}) \cap D = \emptyset$.

From the claim we obtain $\forall B \in \mathcal{B}_n \setminus \Gamma$: $f[B] \cap D = \emptyset$ and $\forall C \in \mathcal{C}_n \setminus \Gamma$: $g[C] \cap D = \emptyset$. Hence $f^{-1}(D) \cap K = (\bigcup\{ B \mid B \in \mathcal{B}_n \cap \Gamma \}) \cap K$ and $g^{-1}(D) \cap L = (\bigcup\{ C \mid C \in \mathcal{C}_n \cap L \}) \cap L$. Therefore $(\mathcal{M}_1(f))(\beta)(D) = \beta(f^{-1}(D)) = \beta(\bigcup\{ B \mid B \in \mathcal{B}_n \cap \Gamma \}) = \sum\{ \beta(B) \mid B \in \mathcal{B}_n \cap \Gamma \}$ and $(\mathcal{M}_1(g))(\gamma)(D) = \gamma(g^{-1}(D)) = \gamma(\bigcup\{ C \mid C \in \mathcal{C}_n \cap \Gamma \}) = \sum\{ \gamma(C) \mid C \in \mathcal{C}_n \cap \Gamma \}$. As $(\mathcal{M}_1(f)f)(\beta) = (\mathcal{M}_1(g))(\gamma)$, it follows that $(\mathcal{M}_1(f))(\beta)(D) = (\mathcal{M}_1(f))(\gamma)(D)$ and $\sum\{\beta(B) \mid B \in \mathcal{B}_n \cap \Gamma \} = \sum\{ \gamma(C) \mid C \in \mathcal{C}_n \cap \Gamma \}$. Now, for the value of Γ we have, since $src \in \Gamma$, $snk \notin \Gamma$

$$\begin{aligned}
&\textstyle\sum\{ \beta(B) \mid B \in \mathcal{B}_n \setminus \Gamma \} \ + \ \sum\{ \gamma(C) \mid C \in \mathcal{C}_n \cap \Gamma \} \\
&\quad = \textstyle\sum\{ \beta(B) \mid B \in \mathcal{B}_n \setminus \Gamma \} \ + \ \sum\{ \beta(B) \mid B \in \mathcal{B}_n \cap \Gamma \} \\
&\quad = \textstyle\sum\{ \beta(B) \mid B \in \mathcal{B}_n \} \\
&\quad = \beta(\textstyle\bigcup\{ B \mid B \in \mathcal{B}_n \}) \\
&\quad = 1.
\end{aligned}$$

The cut $\{src\}$ has value 1: $\sum\{ \beta(B) \mid B \in \mathcal{B}_n \} = \beta(\bigcup\{ B \mid B \in \mathcal{B}_n \}) = \beta((\bigcup \mathcal{B}_n) \cap K) = \beta(K) = 1$ since $K = spt(\beta)$. It follows that the network Net_n has 1 as its minimal cut value. $\square$

Next we will discuss the construction of the measure α on opens (cf. Definition 4.3). The general idea is to organize the collections MFA_n of all max-flow assignments for the network Net_n in a tree. The n-th level of the tree corresponds with the collection MFA_n. There is a branch from a max-flow assignment α_n at level n to a max-flow assignment α_{n+1} at level $n+1$ if α_{n+1} coincides with α_n on Net_n. See the operation $\alpha[n]$ below. However, in this way the tree is, in general, not finite at each level. For König's Lemma to apply we introduce equivalence relations with finitely many equivalence classes at each level. The equivalences respect the tree structure discussed above. The concrete construction is slightly complicated in order to assure converge of the defining limits in Lemma 4.4.

Let, for $n \in \mathbb{N}$, MFA_n denote the collection of all max-flow assignments for Net_n. For $\alpha \in MFA_n$ we have

$$\forall B \in \mathcal{B}_n \colon \alpha(src, B) = \beta(B) \text{ and } \forall C \in \mathcal{C}_n \colon \alpha(C, snk) = \gamma(C). \tag{4.1}$$

Put $\alpha(B,C)=0$ for $\alpha \in MFA_n$ if $B \in \mathcal{B}_n$, $C \in \mathcal{C}_n$ are such that $(B+2^{-n}) \cap (C+2^{-n}) = \emptyset$. From the construction of Net_n we then obtain, for $B \in \mathcal{B}_n$,

$$\beta(B) = \sum\{\, \alpha(B,C) \mid C \in \mathcal{C}_n \,\} \tag{4.2}$$

and, similarly, for $C \in \mathcal{C}_n$,

$$\gamma(C) = \sum\{\, \alpha(B,C) \mid B \in \mathcal{B}_n \,\}. \tag{4.3}$$

Conversely, if a flow assignment α satisfies conditions (4.1) thru (4.3), then from the structure of Net_n it follows that α is a max-flow assignment. If $m \geq n$ and $\alpha \in MFA_m$ we define the flow assignment $\alpha[n]$ by $\alpha[n](src, B) = \beta(B)$ for $B \in \mathcal{B}_n$, $\alpha[n](C, snk) = \gamma(C)$ for $C \in \mathcal{C}_n$ and for an edge (B,C) in Net_n

$$\alpha[n](B,C) = \sum\{\, \alpha(B',C') \mid B' \in \mathcal{B}_m, C' \in \mathcal{C}_m \colon B' \times C' \subseteq B \times C \,\}.$$

Then one easily calculates that $\alpha[n] \in MFA_n$:

$$\begin{aligned}
&\beta(B)\\
&\quad = [\text{property measure } \beta] \;\; \textstyle\sum\{\, \beta(B') \mid B' \in \mathcal{B}_m \colon B' \subseteq B \,\}\\
&\quad = [\text{equation (4.2)}, \ \alpha \in MFA_m] \;\; \textstyle\sum\{\, \sum\{\, \alpha(B',C') \mid C' \in \mathcal{C}_m \,\} \mid B' \in \mathcal{B}_m \colon B' \subseteq B \,\}\\
&\quad = [\text{all } B' \times C' \text{ disjoint}]\\
&\qquad \textstyle\sum\{\, \sum\{\, \alpha(B',C') \mid B' \in \mathcal{B}_m, C' \in \mathcal{C}_m \colon B' \times C' \subseteq B \times C \,\} \mid C \in \mathcal{C}_n \,\}\\
&\quad = [\text{definition } \alpha[n]] \;\; \textstyle\sum\{\, \alpha[n](B,C) \mid C \in \mathcal{C}_n \,\}
\end{aligned}$$

and, similarly,

$$\gamma(C) = \sum\{\, \alpha[n](B,C) \mid B \in \mathcal{B}_n \,\}$$

for all $C \in \mathcal{C}_n$. It follows that $\alpha[n] \in MFA_n$.

Let $n, r \in \mathbb{N}$. Define the binary relation $\sim_r$ on MFA_n by

$$\alpha_1 \sim_r \alpha_2 \iff \forall B \in \mathcal{B}_n \forall C \in \mathcal{C}_n \exists k \in \mathbb{N} \colon (k-1) \cdot 2^{-r} \leq \alpha_1(B,C), \alpha_2(B,C) < k \cdot 2^{-r},$$

for $\alpha_1, \alpha_2 \in MFA_n$, i.e. the binary development of the $\alpha_1(B,C)$'s and $\alpha_2(B,C)$'s are pairwise equal up to the r-th digit. Clearly $\sim_r$ is an equivalence relation on MFA_n and, since the α's are max-flow assignments of the finite network Net_n, $\sim_r$ has finitely many equivalence classes.

Fix an arbitrary sequence $(\alpha_n)_n$ through $(MFA_n)_n$. We construct a sequence of indices $(n_p)_p$, a decreasing sequence of index sets $(N_p)_p$ and a sequence $(r(p))_p$ in $\mathbb{N}$ as follows: Put $N_{-1} = \mathbb{N}$. Let

- $n_p = \min(N_{p-1})$
- $r(p)$ be such that $\kappa(n_p) \cdot \lambda(n_p) \cdot 2^{-r(p)} < 2^{-p}$
- $N_p \subseteq N_{p-1} \setminus \{n_p\}$ be infinite such that $\forall q_1, q_2 \in N_p \colon \alpha_{q_1}[n_p] \sim_{r(p)} \alpha_{q_2}[n_p]$.

This can be achieved since each MFA_n is nonempty, $\alpha_m[n] \in MFA_n$ for $m \geq n$, and $\sim_r$ has finitely many equivalence classes.

We are now in a position to define the measure α on the opens of $\mathcal{B} \times \mathcal{C}$.

Definition 4.3 Let the sequence $(n_p)_p$ be as chosen above. Define $\alpha\colon \mathcal{O}_* \to [0,1]$ by

$$\alpha(O) = \lim_{p\to\infty} \bar{\alpha}_{n_p}(O),$$

for $O \in \mathcal{O}_*$, where $\bar{\alpha}_{n_p}(O) \stackrel{df}{=} \sum\{\, \alpha_{n_p}(B,C) \mid B \in \mathcal{B}_{n_p}, C \in \mathcal{C}_{n_p} \colon B \times C \subseteq O \,\}$.

First we check that the limit at the right-hand side in Definition 4.3 exists.

Lemma 4.4 *The function $\alpha\colon \mathcal{O}_* \to [0,1]$ is well-defined.*

Proof Pick $O \in \mathcal{O}_*$. We check that $(\bar{\alpha}_{n_p}(O))_p$ is a Cauchy-sequence. Let $\varepsilon > 0$. Pick p such that $O \in \mathcal{O}_{n_p}$ and $2^{-p} < \varepsilon$. Then, for $q_1, q_2 > p$ we have $n_{q_1}, n_{q_2} \in N_p$ and $\alpha_{n_{q_1}}[n_p] \sim_{r(p)} \alpha_{n_{q_2}}[n_p]$ in MFA_{n_p}. In general, for $q > p$, it follows from $O \in \mathcal{O}_{n_p}$ that

$$\begin{aligned}
&\bar{\alpha}_{n_q}(O)\\
&\quad = [\text{definition } \bar{\alpha}_{n_q}] \;\; \textstyle\sum\{\, \alpha_{n_q}(B',C') \mid B' \in \mathcal{B}_{n_q}, C' \in \mathcal{C}_{n_q} \colon B' \times C' \subseteq O \,\}\\
&\quad = [\text{ultrametricity}]\\
&\qquad \textstyle\sum\{\, \sum\{\, \alpha_{n_q}(B',C') \mid B' \in \mathcal{B}_{n_q}, C' \in \mathcal{C}_{n_q} \colon B' \times C' \subseteq B \times C \,\} \mid\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad B \in \mathcal{B}_{n_p}, C \in \mathcal{C}_{n_p} \colon B \times C \subseteq O \,\}\\
&\quad = [\text{definition } \alpha_{n_q}[n_p]] \;\; \textstyle\sum\{\, \alpha_{n_q}[n_p](B,C) \mid B \in \mathcal{B}_{n_p}, C \in \mathcal{C}_{n_p} \colon B \times C \subseteq O \,\}
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&|\bar{\alpha}_{n_{q_1}}(O) - \bar{\alpha}_{n_{q_2}}(O)|\\
&\quad \leq \textstyle\sum\{\, |\alpha_{n_{q_1}}[n_p](B,C) - \alpha_{n_{q_2}}[n_p](B,C)| \mid B \in \mathcal{B}_{n_p}, C \in \mathcal{C}_{n_p} \colon B \times C \subseteq O \,\}\\
&\quad \leq [\alpha_{n_{q_1}}[n_p] \sim_{r(p)} \alpha_{n_{q_2}}[n_p]] \;\; \kappa(n_p) \cdot \lambda(n_p) \cdot 2^{-r(p)}\\
&\quad < [\text{definition } r(p)] \;\; 2^{-p}\\
&\quad < \varepsilon.
\end{aligned}$$

So the sequence $(\bar{\alpha}_{n_p}(O))_p$ is Cauchy and $\alpha(O) = \lim_{p\to\infty} \alpha_{n_p}(O)$ is well-defined. □

Most of the work of finding a measure $\alpha \in \mathcal{M}_1(\mathcal{A})$ such that $\mathcal{M}_1(\pi_1)(\alpha) = \beta$ and $\mathcal{M}_1(\pi_2)(\alpha) = \gamma$ has been achieved by now. We will subsequently verify the various properties for α, viz. σ-additivity and the compactness of its support. First we need a technical property.

Lemma 4.5 *The function $\alpha\colon \mathcal{O}_* \to [0,1]$ vanishes outside $K \times L$, i.e. $\alpha(O) = 0$, for $O \in \mathcal{O}_*$, if $O \cap (K \times L) = \emptyset$.*

Proof Let $O \in \mathcal{O}_*$ such that $O \cap (K \times L) = \emptyset$. If $O \in \mathcal{O}_n$ then also $O \cap ((K \times L) + 2^{-n}) = \emptyset$, hence $(B \times C) \cap O = \emptyset$ for $B \in \mathcal{B}_n$, $C \in \mathcal{C}_n$ as $B \times C \subseteq (K \times L) + 2^{-n}$. Therefore, for p sufficiently large,

$$\bar{\alpha}_{n_p}(O) = \sum\{\, \alpha_{n_p}(B,C) \mid B \in \mathcal{B}_{n_p}, C \in \mathcal{C}_{n_p} \colon B \times C \subseteq O \,\} = \sum \emptyset = 0.$$

It follows that $\alpha(O) = \lim_{p\to\infty} \bar{\alpha}_{n_p}(O) = 0$ and that α vanishes outside $K \times L$. □

We check that, with an appeal to Theorem 3.1, the measure α on $\mathcal{O}_*$ will induce a Borel probability measure on $\mathcal{A}$.

Lemma 4.6 *The function $\alpha\colon \mathcal{O}_* \to [0,1]$ is a measure with $\alpha(\mathcal{B} \times \mathcal{C}) = 1$.*

Proof Note that, by definition of $\bar{\alpha}_{n_p}$ (all $p \in \mathbb{N}$), it holds that α is finitely additive. Let $\{O_i \mid i \in I\}$ be a countable and disjoint subcollection of $\mathcal{O}_*$. Suppose $O = \bigcup\{O_i \mid i \in I\} \in \mathcal{O}_*$. We check that $\alpha(O) = \sum\{\, \alpha(O_i) \mid i \in I \,\}$.

Put $O' = (\mathcal{B} \times \mathcal{C}) \setminus O$. Since $O \in \mathcal{O}_*$ we have, by ultrametricity $O' \in \mathcal{O}_*$. Now $\{O'\} \cup \{O_i \mid i \in I\,\}$ is an open cover of the compactum $K \times L$. Let $I_0 \subseteq I$ be finite and minimal such that $K \times L \subseteq \bigcup\{\, O_i \in i \in I_0 \,\}$. Note, by minimality of I_0, $O_i \cap (K \times l) \neq \emptyset \iff i \in I_0$. We then have

$$\begin{aligned}
&\textstyle\sum_{i \in I} \alpha(O_i) \\
&\quad = [\alpha \text{ vanishes outside } K \times L] \;\; \textstyle\sum\{\, \alpha(O_i) \mid i \in I_0 \,\} \\
&\quad = [\text{finite additivity}] \;\; \alpha(\bigcup\{\, O_i \mid i \in I_0 \,\}) + \alpha(\bigcup\{\, O_i \mid i \in I \setminus I_0 \,\}) \\
&\quad = [\text{finite additivity}] \;\; \alpha(\bigcup\{\, O_i \mid i \in I \,\}).
\end{aligned}$$

So, α is σ-additive. For any $p \in \mathbb{N}$,

$$\begin{aligned}
&\bar{\alpha}_{n_p}(\mathcal{B} \times \mathcal{C}) \\
&\quad = \textstyle\sum\{\, \alpha_{n_p}(B, C) \mid B \in \mathcal{B}_n, C \in \mathcal{C}_n \,\} \\
&\quad = [\text{equation (4.2)}] \;\; \textstyle\sum\{\, \beta(B) \mid B \in \mathcal{B}_n \,\} \\
&\quad = [K \subseteq \bigcup \mathcal{B}_n] \;\; 1.
\end{aligned}$$

By definition of α it follows that $\alpha(\mathcal{B} \times \mathcal{C}) = 1$. $\square$

Next we establish that the measure α in fact lives on $\mathcal{A}$ and that it has a compact support.

Lemma 4.7 *The measure $\alpha\colon \mathcal{O}_* \to [0,1]$ has a compact support H with $H \subseteq \mathcal{A}$.*

Proof Put $H = (K \times L) \cap \mathcal{A}$. Then H is compact in $\mathcal{B} \times \mathcal{C}$ as intersection of a compactum and a closed set, hence H is compact in $\mathcal{A}$. Suppose $O \in \mathcal{O}_n$ satisfies $O \cap H = \emptyset$. Note $d(O, H) \geq 2^{-n}$. Let $m \geq n$. Then, for all $B \in \mathcal{B}_m$, $C \in \mathcal{C}_m$ such that $f[B]+2^{-m} = g[C]+2^{-m}$, we have $B \times C \subseteq (K \times L) + 2^{-m}$, $(B \times C) \cap (\mathcal{A} + 2^{-m}) \neq \emptyset$ and $B \times C \subseteq H + 2^{-m}$. Thus $(B \times C) \cap O = \emptyset$. Therefore, $\bar{\alpha}_{n_p}(O) = 0$ for p sufficiently large and $\alpha(O) = 0$ follows. Hence H is compact in $\mathcal{A}$ and α has a compact support. $\square$

By now we have collected sufficiently many results to prove that a measure α in $\mathcal{M}_1(\mathcal{A})$ exists which projects to β in $\mathcal{M}_1(\mathcal{B})$ and to γ in $\mathcal{M}_1(\mathcal{C})$.

Lemma 4.8 *Let $\alpha \in \mathcal{M}_1(\mathcal{A})$ be the Borel probability measure induced by α on $\mathcal{O}_*$. Then $(\mathcal{M}_1(\pi_1))(\alpha) = \beta$ and $(\mathcal{M}_1(\pi_2)(\alpha) = \gamma$.*

Proof By the previous lemma, the measure α is well-defined: The measure α on $\mathcal{O}_* = \mathcal{O}_*(\mathcal{B}\times\mathcal{C})$ induces, by Theorem 3.1, a Borel probability measure on $Bor(\mathcal{B}\times\mathcal{C})$. By Lemma 4.7, $\alpha(E) = 0$ for each $E \in Bor(\mathcal{B} \times \mathcal{C})$ such that $E \cap \mathcal{A} = \emptyset$. Hence, α on $Bor(\mathcal{B} \times \mathcal{C})$ restricts to a Borel probability measure α on $Bor(\mathcal{A})$.

Suppose $B \in \mathcal{B}_n$. Then $(\mathcal{M}_1(\pi_1))(\alpha)(B) = \alpha(\pi_1^{-1}(B)) = \alpha(B \times \mathcal{C}) = \sum\{ \alpha(B, C) \mid C \in \mathcal{C}_n \} = \beta(B)$ by construction of α, Lemma 4.5 and equation (4.2). It follows that $(\mathcal{M}_1(\pi_1))(\alpha)$ and β coincide on $\mathcal{O}_*$, hence $(\mathcal{M}_1(\pi_1))(\alpha) = \beta$.

Summarizing this section we have the following result.

Theorem 4.9 *Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be ultrametric spaces and $f: B \to D$, $g: C \to D$ be nonexpansive mappings. Let $(\mathcal{A}, \pi_1, \pi_2)$ be the pullback of f and g in* **UMS**. *Suppose $\beta \in \mathcal{M}_1(\mathcal{B})$ and $\gamma \in \mathcal{M}_1(\mathcal{C})$ satisfy $\mathcal{M}_1(f)(\beta) = \mathcal{M}_1(g)(\gamma)$. Then there exists $\alpha \in \mathcal{M}_1(\mathcal{A})$ such that $\mathcal{M}_1(\pi_1)(\alpha) = \beta$ and $\mathcal{M}_1(\pi_2)(\alpha) = \gamma$.* □

5 Preservation of weak pullbacks

The functor $\mathcal{M}_1$ on **UMS** possesses several attractive properties from the coalgebraic point of view, e.g., preservation of completeness and local nonexpansiveness (cf. [VR97a]). Concerning the preservation by $\mathcal{M}_1$ of weak pullbacks Theorem 4.9 of the previous section allows us to prove that $\mathcal{M}_1$ preserves weak pullbacks as functor from **UMS** to **SET**. However, so far we are not able to prove that $\mathcal{M}_1$ does preserve weak pullbacks as functor on **UMS**, as the construction of Section 4 does not guarantee the nonexpansiveness of the mediating arrow. In this section we present the positive result for **SET** and discuss a possible way to remedy the situation for **UMS**.

Theorem 5.1 *Let the functor $\mathcal{M}_1$:* **UMS** $\to$ **SET** *be the composition of $\mathcal{M}_1$ on* **UMS** *of Definition 4.1 and the forgetful functor from* **UMS** *to* **SET**. *Then $\mathcal{M}_1$ from* **UMS** *to* **SET** *preserves weak pullbacks.*

Proof By general coalgebraic considerations it suffices to show that $\mathcal{M}_1$ transforms pullbacks in **UMS** into weak pullbacks in **SET**.

Suppose $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and $f: \mathcal{B} \to \mathcal{D}$, $g: \mathcal{C} \to \mathcal{D}$ are objects and arrows in **UMS**. Let $(\mathcal{A}, \pi_1, \pi_2)$ be the pullback of f and g in **UMS**. By functoriallity, $\mathcal{M}_1(\pi_1): \mathcal{M}_1(\mathcal{A}) \to \mathcal{M}_1(\mathcal{B})$, $\mathcal{M}_1(\pi_2): \mathcal{A} \to \mathcal{M}_1(\mathcal{C})$ and $\mathcal{M}_1(f) \circ \mathcal{M}_1(\pi_1) = \mathcal{M}_1(g) \circ \mathcal{M}_1(\pi_2)$ in **SET**.

Suppose $\mathcal{X}$ and $\phi: \mathcal{X} \to \mathcal{M}_1(\mathcal{B})$, $\psi: \mathcal{X} \to \mathcal{M}_1(\mathcal{C})$ are objects and arrows in **SET** such that $\mathcal{M}_1(f) \circ \phi = \mathcal{M}_1(g) \circ \psi$. Choose a mapping $\chi: \mathcal{X} \to \mathcal{M}_1(\mathcal{A})$ such that $\mathcal{M}_1(\pi_1)(\chi(x)) = \phi(x)$ and $\mathcal{M}_1(\pi_2)(\chi(x)) = \psi(x)$ for all $x \in \mathcal{X}$. Such a mapping exists by Theorem 4.9. By construction we then have $\mathcal{M}_1(\pi_1) \circ \chi = \phi$ and $\mathcal{M}_1(\pi_2) \circ \chi = \psi$. Hence, χ is a mediating function for $(\mathcal{X}, \phi, \psi)$ to $(\mathcal{M}_1(\mathcal{A}), \mathcal{M}_1(\pi_1), \mathcal{M}_1(\pi_2))$. Conclusion: $(\mathcal{M}_1(\mathcal{A}), \mathcal{M}_1(\pi_1), \mathcal{M}_1(\pi_2))$ is a weak pullback for $\mathcal{M}_1(f)$ and $\mathcal{M}_1(g)$ in **SET**. □

The proof of Theorem 5.1 would carry over to a proof of the preservation of weak pullbacks by $\mathcal{M}_1$ on **UMS** is the arrow χ constructed above would be nonexpansive. However, the construction of the α's in the previous section gives no clue about the distance $d_m(\alpha_1, \alpha_2)$ of two Borel probability measures α_1, α_2 obtained for pairs of measures β_1, γ_1 and β_2, γ_2 such that $d_m(\beta_1, \beta_2), d_m(\gamma_1, \gamma_2) < 2^{-n}$ for some $n \in \mathbb{N}$. Intuitively speaking, the construction is not uniform in β and γ.

If $d_m(\beta_1, \beta_2), d_m(\gamma_1, \gamma_2) < 2^{-n}$ then, by definition of d_m, the measures β_1, β_2 coincide on $\mathcal{O}_n(\mathcal{B})$ and γ_1, γ_2 coincide on $\mathcal{O}_n(\mathcal{C})$. Therefore, the corresponding networks Net_n for β_1, γ_1 and Net_n for β_2, γ_2 are the same: It is straightforward to show that $d(K_1, K_2) < 2^{-n}$ for the supports K_1, K_2 of β_1, β_2, respectively. Similarly $d(L_1, L_2) < 2^{-n}$ for the supports L_1, L_2 of the measures γ_1 and γ_2. Hence, by ultrametricity, the collections of covering balls $\mathcal{B}_n$ and $\mathcal{C}_n$ are equal for β_1, γ_1 and β_2, γ_2. Thus the networks have the same nodes. The capacities of the source and sink edges also coincide since they depend on the values $\beta_1(B) = \beta_2(B)$ for $B \in \mathcal{B}_n$, and on $\gamma_1(C) = \gamma_2(C)$ for $C \in \mathcal{C}_n$. The remaining edges are determined by f and g, independently of β_1, γ_1 and β_2, γ_2.

Clearly, we can choose uniformly a max-flow assignment for each network Net_n. A natural candidate is the max-flow assignment α_n given by

$$\begin{aligned} \alpha_n(B, C) &= \frac{\sum\{\, \beta(B') \mid f[B'] \equiv_n f[B] \,\}}{\sharp\{\, (B', C') \mid f[B'] \equiv_n f[B] \wedge g[C] \equiv_n g[C] \,\}} \\ &= \frac{\sum\{\, \gamma(C') \mid g[C'] \equiv_n g[C] \,\}}{\sharp\{\, (B', C') \mid f[B'] \equiv_n f[B] \wedge g[C] \equiv_n g[C] \,\}} \end{aligned}$$

where B, B' range over $\mathcal{B}_n$ and C, C' range over $\mathcal{C}_n$ and, for $D, D' \subseteq \mathcal{D}$, $D \equiv_n D'$ iff $D + 2^{-n} = D' + 2^{-n}$. Thus we simply can distributed the total sum $\sum\{\, \beta(B') \mid f[B'] \equiv_n f[B] \,\}$ over all squares (B', C') that map into the same 2^{-n}-ball in $\mathcal{D}$ as the square (B, C); each square gets an average share.[1] However, we do not know if the definition

$$\alpha(O) \quad = \quad \lim_{n \to \infty} \bar{\alpha}_n(O) \tag{5.4}$$

would work out. In particular, we are not convinced that the sequence $\bar{\alpha}_n(B, C))_n$ converges. It is to be expected that further combinatorial analysis will be needed to clarify this matter. We conjecture that, in case the equation (5.4) is well-defined, the functor $\mathcal{M}_1$ preserves weak pullbacks on **UMS**.

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[1]Interestingly, the networks without source and sink are disjoint sums of complete bipartite graphs, which corresponds to the notion of saturation coined in [VR97a].)

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