A note on the completeness of $\mathcal{M}_1(X)$

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Abstract For an ultra-metric space X the collection $\mathcal{M}_1(X)$ of Borel probability measures of compact support is an ultra-metric space when endowed with metric based on the agreement on a special kind of open subsets. This note addresses the issue of preservation of completeness of the space $\mathcal{M}_1(X)$ given the completeness of the space X.

Introduction

The functor $\mathcal{M}_1$ was introduced in [VR97, VR99] for the co-algebraic description of probabilistic transition systems and their bisimulation as extension of the work of Larsen and Skou [LS91] (see also [BDEP97]). Working on the category of ultra-metric spaces and non-expansive functions, $\mathcal{M}_1$ maps an ultra-metric space X to the ultra-metric space $\mathcal{M}_1(X)$ of all Borel probability measures on X of compact support and $\mathcal{M}_1$ maps a function f to the 'measures along f' mapping $\lambda\mu.\mu \circ f^{-1}$. In [Vin98] the $\mathcal{M}_1$ functor is further studied, in particular regarding its preservation of weak pullbacks. In the work mentioned no proof has been provided of $\mathcal{M}_1$ preserving completeness. This note aims to overcome this.

A completeness result

For the remainder of the paper we fix an ultra-metric space X endowed with a metric d. We assume familiarity with the basic definitions and result of metric topology and measure theory. The reader may consult [Dug66, Eng89, Smy92, BV96] and [Hal50, Rud66] for further details. We use $\mathcal{O}$ to denote the collection of open subsets of X. An open subset O is said to be 2^{-n}-closed if $\forall x \in O{:}B_{2^{-n}}(x) \subseteq O$. We put $\mathcal{O}_n = \{\, O \mid O \text{ } 2^{-n}\text{-closed} \,\}$ and $\mathcal{O}_* = \bigcup_n \mathcal{O}_n$.

Below we will apply the following standard result from metric semantics (see [BZ82, BV96]).

Proposition 1 *If the ultra-metric space X is complete, then so is the ultra-metric space $\mathcal{P}_{co}(X)$.*

A Borel probability measure μ on X is said to be of compact support if there exists a compact set $spt(\mu)$ such that, for any $O \in \mathcal{O}(X)$, $\mu(O) = 0 \iff O \cap spt(\mu) = \emptyset$. It holds that $spt(\mu)$ is the minimal compact set of measure 1 (cf. [Rud66]). We use $\mathcal{M}_1(X)$ to denote the collection of Borel probability measures on X of compact support. We define a metric d on $\mathcal{M}_1(X)$ as follows:

$$d(\mu,\nu) = \inf\{\, 2^{-n} \mid \forall O \in \mathcal{O}_n{:}\mu(O) = \nu(O) \,\}.$$

It is easy to check most properties for a metric. However, to prove that $\mu = \nu$ if $\forall O \in \mathcal{O}_*{:}\mu(O) = \nu(O)$ we need a basic result from measure theory.

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Lemma 2 *Every measure μ on $\mathcal{O}_*$ has a unique extension $\bar{\mu}$ to $\mathcal{B}$.*

Proof We have that μ is a measure on the ring $\mathcal{O}_*$. The σ-ring generated by $\mathcal{O}_*$ coincides with the Borel-σ-algebra $\mathcal{B}$. By [Hal50, Theorem 13.A] there exists a unique extension $\bar{\mu}$ of μ to the σ-ring generated by $\mathcal{O}_*$, i.e., there a unique extension $\bar{\mu}$ of μ to $\mathcal{B}$. $\square$

Before addressing the completeness of $\mathcal{M}_1(X)$ we first state some useful properties of the Hausdorff-distance and the distance on measures.

Property 3

(a) *For $E, F \subseteq X$ it holds that $d(E,F) \leq 2^{-n}$ iff $\forall O \in \mathcal{O}_n \colon E \cap O = \emptyset \iff F \cap O = \emptyset$.*

(b) *For $\mu, \nu \in \mathcal{M}_1(X)$ it holds, $d(spt(\mu), spt(\nu)) \leq 2^{-n}$ if $d(\mu,\nu) \leq 2^{-n}$*

Proof Part (a) follows from the definition of the Hausdorff-distance on $\mathcal{P}(X)$. Part (b) is straightforward using part (a). $\square$

We have now arrived at the main result of this note.

Theorem 4 *If X is complete, then $\mathcal{M}_1(X)$ is complete.*

Proof Suppose that X is an ultra-metric space and let $(\mu_i)_i$ be a Cauchy sequence in $\mathcal{M}_1(X)$. Let the index i_n be such that $\forall i, j > i_n \colon \mu_i(O) = \mu_j(O)$ for $O \in \mathcal{O}_n$. Define the mapping $\mu \colon \mathcal{O}_* \to [0,1]$ by $\mu(O) = \lim_i \mu_i(O)$.

Claim 1: μ is finitely additive. Proof of Claim 1: Supose $O_1, \ldots, O_r \in \mathcal{O}_*$ are disjunct. Pick n such that $O_1, \ldots, O_r \in \mathcal{O}_n$. Note $O_1 \cup \ldots \cup O_r \in \mathcal{O}_n$. Pick $i > i_n$. Then $\mu(O_1 \cup \ldots \cup O_r) = \mu_i(O_1 \cup \ldots \cup O_r) = \mu_i(O_1) + \cdots + \mu_i(O_r) = \mu(O_1) + \cdots + \mu(O_r)$.

Claim 2: μ has compact support. Proof of claim 2: Put $K_i = spt(\mu_i)$. By Property 3b it holds that $(K_i)_i$ is a Cauchy sequence in the hyperspace $\mathcal{P}_{co}(X)$ of compact subsets of X. By completeness of X, the hyperspace $\mathcal{P}_{co}(X)$ is complete (cf. [BV96]). Therefore $K = \lim_i K_i$ exists and is compact. Let $O \in \mathcal{O}_n$. Pick $i > i_n$. Then

$$
\begin{aligned}
& O \cap K = \emptyset \\
\iff\ & [\text{property 3a}] \;\; O \cap K_i = \emptyset \\
\iff\ & [K_i \text{ support of } \mu_i] \;\; \mu_i(O) = 0 \\
\iff\ & [i > i_n] \;\; \mu(O) = 0.
\end{aligned}
$$

So, $K = spt(\mu)$ and K is compact.

Claim 3: μ is σ-additive. Proof of claim 3: Let $(O_i)_i$ be a disjoint sequence in $\mathcal{O}_*$ such that $\bigcup_i O_i \in \mathcal{O}_*$. Pick n such that $\bigcup_i O_i \in \mathcal{O}_n$. Put $O' = X \setminus (\bigcup_i O_i)$. Note that $O' \in \mathcal{O}_n$. In particular is O' an open set. We have that $\{ O_i \mid i \in \mathbb{N} \} \cup \{O'\}$ is an open cover of K. Since K is compact, we derive that the index-set $I = \{ i \mid O_i \cap K \neq \emptyset \}$ is finite. Note, by disjointness, $(\bigcup_{i \notin I} O_i) \cap K = \emptyset$, hence $\mu(\bigcup_{i \notin I} O_i) = 0$. For $i \notin I$ we have $\mu(O_i) = 0$. Therefore $\mu(\bigcup_{i \notin I} O_i) = \sum_{i \notin I} \mu(O_i)$. We conclude

$$
\begin{aligned}
& \mu(\textstyle\bigcup_i O_i) \\
=\ & \mu(\textstyle\bigcup_{i \in I} O_i \cup \bigcup_{i \notin I} O_i)
\end{aligned}
$$

$$= \text{[finite additivity]} \;\; \sum_{i\in I} \mu(O_i) + \mu(\bigcup_{i\notin I} O_i)$$

$$= \sum_{i\in I} \mu(O_i) \sum_{i\notin I} \mu(O_i)$$

$$= \sum_i \mu(O_i).$$

Define $\mu: \mathcal{B} \to [0,1]$, using Lemma 2, as the unqiue extension of μ on $\mathcal{O}_*$. $\square$

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