On Relational Properties of Lumpability

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Abstract—**The technique of lumping of Markov chains is one of the main tools for recovering from state explosion in the quantitative analysis of large probabilistic systems. Useful results regarding relational properties of general, ordinary and exact lumpability, in particular transitivity and strict confluence, are rather scattered or implicit in the literature. These are collected and reformulated here from a process-theoretic point of view. Additionally, counterexamples are provided that complete the picture.**

Keywords—**Probabilistic systems, general lumpability, ordinary and exact lumpability, transitivity, strict confluence**

I. INTRODUCTION

The PROGRESS project (a)MaPAoTS focusses on the modelling and performance analysis of larger telecommunication and network systems. Central in this research is the system specification and analysis language POOSL ([Voe94], [PV97]) as been developed by Voeten and Van der Putten. Using POOSL and the simulation toolkit based on it, large systems with probabilistic and timed behavior have been modelled and evaluated, see e.g. [TVP$^+$99], [TVB$^+$01], [HVT02], [HVP$^+$02]. The models arising in these industrial case studies suffer from state space explosion problem, i.e., the set of states of the models is too large to fit in the simulation tool. Therefore, reduction techniques are needed that transform the systems at hand into smaller, necessary more abstract systems, while the characteristic behavior and performance metrics remain. Since the models under consideration are in essence Markov chains, we study notions and techniques related to the reduction of the state space of Markov chains. Some such techniques were proposed in [PVT01] and [BVP01]. Lumpability of Markov chains is another reduction technique from the theory of Markov chains. In this paper we study lumpability and two its variants together with relational properties that are relevant in the context of the project.

The notion of general lumpability can be defined quite naturally. States of the one Markov chain can be identified into a single state of the other Markov chain. Global conditions ensure that the probabilistic behavior of the two chains is essentially the same: it is the same for the initial distributions; it remains the same for any arbitrary number of steps in the respective chains.

What makes the notion of general lumpability unattractive from a practical point of view is the infinite nature of the global conditions. In order to verify that one chain can be lumped to another according to this definition an infinite number of equations should be checked. Although structural and inductive arguments may apply, such is undesirable for computational reasons. Therefore, two special instances of general lumpability have been proposed in the literature, viz. that of ordinary lumpability [KS76] and that of exact lumpability [Sch84]. Lumpability was also studied in [SR89] and [Buc94a]. In the context of stochastic process algebra, ordinary lumpability appears in [Hil95], [Buc94b], [BB01], as well as in several survey papers collected in [BHK00]. The applicability of exact lumpability for stochastic process algebra was first exploited in [Buc94b]. In this paper we collect some relevant facts for these notions, in particular relating to transitivity and strict confluence.

When reducing larger systems to smaller ones, one prefers to have transitivity. This means that the abstraction obtained after a number of behaviour and performance preserving steps, probably based on time and memory considerations, is still a correct abstraction of the orginal system. One does not want the essentials to have been vanished during the processes of iterated lumping. The notion of strict confluence is important when the analysis focuses on different aspects of the system that later are combined into a single abstraction. As the efforts spent on the earlier analysis should remain valuable, it need to be possible to reconcile the intermediate models into a single common lumping.

In this paper we focus on the properties of transitivity and strict confluence for general lumping and for ordinary and exact lumping. A better understanding of these notion is pivotal for further improvement of the toolkit supporting POOSL. It also helps in linking the concrete ideas and heuristics learned from the many case studies conducted within the project with existing algebraic process theory,

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enabling the transfer of results and techniques available there to the (a)MaPAoTS setting.

II. Preliminaries

A Markov chain M will be represented as a triple (S, P, π) where S is the finite set of states of M, P the transition probability matrix indicating the probability of getting from one state to another, and π is the initial probability distribution representing the likelyhood for the system to start in a certain state. For more details on finite Markov chains we refer to [How71], [KS76], [Tij94].

Let $f\colon X \to Y$ and $h\colon X \to Z$ be two mappings. The equivalence relation $\sim$ induced by f and h on X is given by

$$x \sim x' \iff \exists n \exists x_0, \ldots, x_n \in X\colon x_0 = x \wedge x_n = x' \wedge \\ \forall i < n\colon f(x_i) = f(x_{i+1}) \vee h(x_i) = h(x_{i+1}).$$

For $x \in X$ its equivalence class with respect to $\sim$ is denoted by $[x]$. For any $x \in X$ and $y \in Y$ we have, by the definition of $\sim$, that either $f^{-1}(y) \subseteq [x]$ or $f^{-1}(y) \cap [x] = \emptyset$. Hence, the class $[x]$ is a disjoint union over $\{ f^{-1}(y) \mid y \in Y, f^{-1}(y) \subseteq [x] \}$. Similarly for h.

III. Lumpability

The concept of lumpability captures the idea of aggregating several states into a single one. As the state space will become smaller, performance analysis (calculation of the steady state distribution, the average throughput, etc.) will be easier. However, the behavior of the Markov chain obtained from the lumping should reflect faithfully that of the original chain. After all, one wants to relate the performance figures computed for the former to performance figures of the latter.

We first present a rather liberal notion of lumping called general lumping.

Definition III.1 Let M_1, M_2 be two Markov chains. A surjective mapping $\ell\colon S_1 \to S_2$ is said to be a general lumping of M_1 to M_2, notation $M_1 \xrightarrow{\ell}_g M_2$, if the following two conditions hold:
(i) $\pi_2(u) = \sum_{s \in \ell^{-1}(u)} \pi_1(s)$ for all $u \in S_2$;
(ii) $\pi_2(u) \cdot P_2^i(u, v) = \sum_{s \in \ell^{-1}(u)} \sum_{t \in \ell^{-1}(v)} \pi_1(s) \cdot P_1^i(s, t)$ for all $u, v \in S_2$ and $i \geq 0$.

Suppose Markov chain M_1 lumps to the Markov chain M_2 via the lumping ℓ, i.e. $M_1 \xrightarrow{\ell}_g M_2$. The first condition states that, with respect to the initail distribution π_1 of M_1 and π_2 of M_2, the total weight of all the states of M_1 that are combined into a single new state in M_2 is the same as the weight of this new state in M_2. The second condition explicitly connects multi-step behavior of M_1 and M_2. The probability of getting in M_2 from a state u to a state v in i-steps is the same as summing up all possible ways of getting in M_1 in i-steps from any state s that is lumped to u to any state t that is lumped to v.

Note that given a Markov chain M_1 and a surjective mapping $\ell{:}S_1 \to S_2$ there is not necessarily a Markov chain M_2 with state set S_2 and a probability matrix P_2 satisfying condition III.1(i) and condition III.1(ii). However, if $M_1 \xrightarrow{\ell}_g M_2$ then the so-called steady state probability vectors $\hat{\pi}_1$ of M_1 and $\hat{\pi}_2$ of M_2 satisfy $\hat{\pi}_2(u) = \sum_{s \in \ell^{-1}(u)} \hat{\pi}_1(s)$, for all $u \in S_2$. This also holds for the transient state probability vectors. As steady state probability vectors and transient state probability vectors are key notions of performance analysis, it follows that lumpability is a useful concept in this setting.

The second condition of Definition III.1 is problematic from a computational point of view as, in principle, infinitely many equations need to be checked. For concrete cases this might be feasible, but no general method is known. The way out here is to refine condition III.1(ii) into a condition that represents one equation only. We discuss below two possible options: ordinary lumpability (Definition III.2) and exact lumpability (Definition III.3).

Definition III.2 Let M_1, M_2 be two Markov chains. A surjective mapping $\ell\colon S_1 \to S_2$ is said to be an ordinary lumping of M_1 to M_2, notation $M_1 \xrightarrow{\ell}_o M_2$, if the following two conditions hold:
(i) $\pi_2(u) = \sum_{s \in \ell^{-1}(u)} \pi_1(s)$ for all $u \in S_2$;
(ii) $P_2(\ell(s), v) = \sum_{t \in \ell^{-1}(v)} P_1(s, t)$ for all $s \in S_1$, $v \in S_2$.

Note that condition (ii) above implies that if two states s, s' of M_1 lump to the same state u of M_2, i.e. $\ell(s) = \ell(s') = u$, then $\sum_{t \in \ell^{-1}(v)} P_1(s, t) = \sum_{t \in \ell^{-1}(v)} P_1(s', t) = P_2(u, v)$ for any state v of M_2.

We check that ordinary lumpability is indeed a special case of general lumpability. The inductive argument for this is based on splitting a sequence of $i + 1$ steps from u to v with probability $P_2^{i+1}(u, v)$ into a first step from u to some u' with probability $P_2(u, u')$ and a sequence of i-steps from u' to v with probability $P_2^i(u', v)$.

Lemma III.3 *If $M_1 \xrightarrow{\ell}_o M_2$ then $M_1 \xrightarrow{\ell}_g M_2$.*

Proof We need to check condition (ii) of Defintion III.1. First we verify $P_2^i(\ell(s), v) = \sum_{t \in \ell^{-1}(v)} P_1^i(s, t)$ for any $s \in S_1$, $v \in S_2$ and $i \geq 0$ by induction on i.

- $[i = 0]$ Straightforward.
- $[i + 1]$ We have for $s \in S_1$, $v \in S_2$, $i \geq 0$ that $\sum_{t \in \ell^{-1}(v)} P_1^{i+1}(s, t)$
$= \sum_{t \in \ell^{-1}(v)} \sum_{s' \in S_1} P_1(s, s') \cdot P_1^i(s', t)$

$=$ [induction hypothesis]
$\sum_{s'\in S_1} P_1(s,s')\cdot P_2^i(\ell(s'),v)$
$=$ $[S_1 = \bigcup\{\ \ell^{-1}(u') \mid u'\in S_2\ \}]$
$\sum_{u'\in S_2}\sum_{s'\in\ell^{-1}(u')} P_1(s,s')\cdot P_2^i(\ell(s'),v)$
$= \sum_{u'\in S_2} P_2^i(u',v)\cdot\sum_{s'\in\ell^{-1}(u')}P_1(s,s')$
$=$ $[M_1 \xrightarrow{\ell}_o M_2]$ $\sum_{u'\in S_2} P_2^i(u',v)\cdot P_2(\ell(s),u')$
$= P_2^{i+1}(\ell(s),v)$.
From the property above we get for $u,v\in S_2$ and $i\geq 0$,
$\pi_2(u)\cdot P_2^i(u,v)$
$=$ [condition III.2(i)] $\sum_{s\in\ell^{-1}(u)}\pi_1(s)\cdot P_2^i(u,v)$
$= \sum_{s\in\ell^{-1}(u)}\sum_{t\in\ell^{-1}(v)}\pi_1(s)\cdot P_1^i(s,t)$
which was to be shown. ■

Next we define the notion of exact lumpability.

Definition III.4 Let M_1, M_2 be two Markov chains. A surjective mapping $\ell\colon S_1\to S_2$ is said to be an exact lumping of M_1 to M_2, notation $M_1\xrightarrow{\ell}_e M_2$, if the following two conditions hold:
(i) $\pi_2(u) = \#\ell^{-1}(u)\cdot\pi_1(s)$ for any $u\in S_2$, $s\in\ell^{-1}(u)$;
(ii) $\sum_{s\in\ell^{-1}(u)} P_1(s,t) = \frac{\#\ell^{-1}(u)}{\#\ell^{-1}(\ell(t))} P_2(u,\ell(t))$ for all $u\in S_2$, $t\in S_1$.

The idea of condition (i) is that states that are lumped into the same new state have equal weight initially. Moreover, condition (ii) implies, for states u,v of M_2,

$$\sum_{s\in\ell^{-1}(u)} P_1(s,t) = \frac{\#\ell^{-1}(u)}{\#\ell^{-1}(v)} P_2(u,v)$$

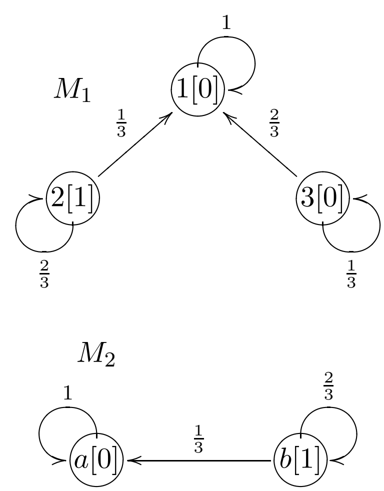
for any t of M_1 such that $\ell(t) = v$. Thus, if $\ell(t) = \ell(t') = v$ then $\sum_{s\in\ell^{-1}(u)} P_1(s,t)$ and $\sum_{s\in\ell^{-1}(u)} P_1(s,t')$ are equal, viz. the same as $\frac{\#\ell^{-1}(u)}{\#\ell^{-1}(v)} P_2(u,v)$.

Lemma III.5 *If $M_1\xrightarrow{\ell}_e M_2$ then $M_1\xrightarrow{\ell}_g M_2$.*

Proof As condition III.1(i) directly follows from condition III.4(i) we only need to check condition III.1(ii). First we prove $P_2^i(u,v) = \frac{\#\ell^{-1}(v)}{\#\ell^{-1}(u)}\sum_{s\in\ell^{-1}(u)} P_1^i(s,t)$ for any $u,v\in S_2$, $t\in\ell^{-1}(v)$ and $i\geq 0$ by induction on i.
- $[i=0]$ Straightforward.
- $[i+1]$ Pick $u,v\in S_2$, $t\in\ell^{-1}(v)$. We then have
$P_2^{i+1}(u,v)$
$= \sum_{v'\in S_2} P_2^i(u,v')\cdot P_2(v',v)$
$=$ [condition III.4(ii)]
$\sum_{v'\in S_2} P_2^i(u,v')\cdot\frac{\#\ell^{-1}(v)}{\#\ell^{-1}(v')}\sum_{t'\in\ell^{-1}(v')}P_1(t',t)$
$=$ [induction hypothesis]
$\sum_{v'\in S_2}\Big(\frac{\#\ell^{-1}(v')}{\#\ell^{-1}(u)}\sum_{s\in\ell^{-1}(u)}P_1^i(s,t'')\Big)\cdot$
$\frac{\#\ell^{-1}(v)}{\#\ell^{-1}(v')}\sum_{t'\in\ell^{-1}(v')}P_1(t',t)$
with $t''\in\ell^{-1}(v')$ arbitrary
$= \frac{\#\ell^{-1}(v)}{\#\ell^{-1}(u)}\sum_{s\in\ell^{-1}(u)}$
$\sum_{v'\in S_2}\sum_{t'\in\ell^{-1}(v')}P_1^i(s,t')\cdot P_1(t',t)$
$=$ $[S_1 = \bigcup\{\ \ell^{-1}(v') \mid v'\in S_2\ \}]$
$\frac{\#\ell^{-1}(v)}{\#\ell^{-1}(u)}\sum_{s\in\ell^{-1}(u)}\sum_{t'\in S_1}P_1^i(s,t')\cdot P_1(t',t)$
$= \frac{\#\ell^{-1}(v)}{\#\ell^{-1}(u)}\sum_{s\in\ell^{-1}(u)}P_1^{i+1}(s,t)$.
From the property above we obtain, for $u,v\in S_2$, $t\in\ell^{-1}(v)$, and $i\geq 0$,
$\pi_2(u)\cdot P_2^i(u,v)$
$= \pi_2(u)\cdot\frac{\#\ell^{-1}(v)}{\#\ell^{-1}(u)}\sum_{s\in\ell^{-1}(u)}P_1^i(s,t)$
$= \frac{\pi_2(u)}{\#\ell^{-1}(u)}\sum_{t'\in\ell^{-1}(v)}\sum_{s\in\ell^{-1}(u)}P_1^i(s,t')$
$=$ [condition III.4(i)]
$\sum_{s\in\ell^{-1}(u)}\sum_{t'\in\ell^{-1}(v)}\pi_1(s)\cdot P_1^i(s,t')$
using that $\sum_{s\in\ell^{-1}(u)}P_1(s,t) = \sum_{s\in\ell^{-1}(u)}P_1(s,t')$ if $\ell(t)=\ell(t')$ for all $\#\ell^{-1}(v)$ elements of $\ell^{-1}(v)$. This was to be shown. ■

An example of a general lumping that is neither ordinary nor exact lumping is shown with the following.

Example Consider the chains M_1 and M_2 given below.



One can easily check that $M_1\xrightarrow{\ell}_g M_2$ with $\ell\colon S_1\to S_2$ given by $\ell(1)=a$, $\ell(2)=\ell(3)=b$ by an inductive argument. However, ℓ does not meet the requirements for an ordinary and an exact lumping. For the ordinary case we note that although 2 and 3 are identified by ℓ we do not have that $P_1(2,1) = P_1(3,1)$. For the exact case we note that $\pi_1(2)\neq\pi_1(3)$.

The example also illustrates the usefulness of general lumping. The state 3 of chain M_1 is an irrelevant part of that chain and can be cut off via the lumping ℓ.

IV. Transitivity

In this section we address the transitivity of the lumpability relation introduced above. Transitivity is not merely of theoretical interest. It justifies repeated lumpings. Suppose we have constructed a sequence of lumping $M_1 \xrightarrow{\ell_1}_g M_2$, $M_2 \xrightarrow{\ell_2}_g M_3$, ..., $M_{n-1} \xrightarrow{\ell_{n-1}}_g M_n$ and that we have calculated some performance measure of M_n (which typically could not be obtained for M_1 though M_{n-1} directly because of memory limitations). Is the computed result relevant for M_1? The transitivity result, Lemma IV.1, implies that we also have $M_1 \xrightarrow{\ell}_g M_n$ where the lumping ℓ is given in terms of $\ell_1, \dots, \ell_{n-1}$. So, every analysis on M_n that is respected by general lumping can be propagated back to M_1. We will show transitivity of general and ordinary lumping and provide a counterexample to the case of exact lumping.

Lemma IV.1 *Let M_1, M_2, M_3 be three Markov chains such that $M_1 \xrightarrow{\ell}_g M_2$ and $M_2 \xrightarrow{k}_g M_3$. Then it holds that $M_1 \xrightarrow{k\circ\ell}_g M_3$.*

Proof We check condition (ii) of Definition III.1. Condition (i) is similar and slightly easier. Pick $w, x \in S_3$ and $i \geq 0$. Note, for $w \in S_3$, we have $(k \circ \ell)^{-1}(w) = \bigcup\{ \ell^{-1}(u) \mid u \in k^{-1}(w) \}$. So,

$\pi_3(w) \cdot P_3^i(w, x)$
$=$ [condition (ii) for $M_2 \xrightarrow{k}_g M_3$]
$\sum_{u\in k^{-1}(w)} \sum_{v\in k^{-1}(x)} \pi_2(u) \cdot P_2^i(u, v)$
$=$ [condition (ii) for $M_1 \xrightarrow{\ell}_g M_2$]
$\sum_{u\in k^{-1}(w)} \sum_{v\in k^{-1}(x)}$
$\quad \sum_{s\in\ell^{-1}(u)} \sum_{t\in\ell^{-1}(v)} \pi_1(s) \cdot P_1^i(s, t)$
$= \sum_{s\in(k\circ\ell)^{-1}(w)} \sum_{t\in(k\circ\ell)^{-1}(x)} \pi_1(s) \cdot P_1^i(s, t)$
which was to be shown. ■

Ordinary lumping is transitive as well as implied by the next lemma.

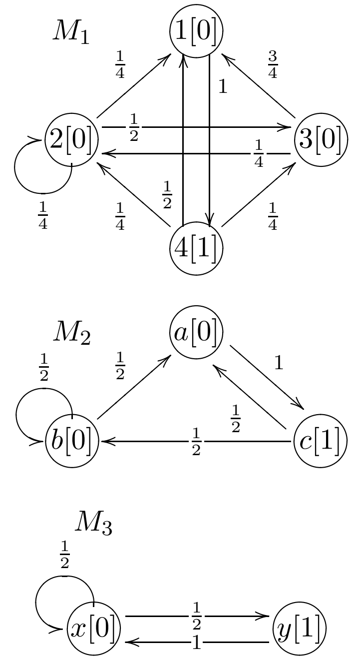
Lemma IV.2 *Let M_1, M_2, M_3 be three Markov chains such that $M_1 \xrightarrow{\ell}_o M_2$ and $M_2 \xrightarrow{k}_o M_3$. Then it holds that $M_1 \xrightarrow{k\circ\ell}_o M_3$.*

Proof We verify condition (ii) of Definition III.2: Pick $s \in S_1$, $x \in S_3$. We then have

$\sum_{t\in(k\circ\ell)^{-1}(x)} P_1(s, t)$
$=$ $[(k \circ \ell)^{-1}(x) = \bigcup\{ \ell^{-1}(v) \mid v \in k^{-1}(x) \}]$
$\sum_{v\in k^{-1}(x)} \sum_{t\in\ell^{-1}(v)} P_1(s, t)$
$=$ $[M_1 \xrightarrow{\ell}_o M_2]$ $\sum_{v\in k^{-1}(x)} P_2(\ell(s), v)$
$=$ $[M_2 \xrightarrow{k}_o M_3]$ $P_3((k \circ \ell)(s), x)$. ■

Exact lumping is not a transtive notion. We provide an counterexample for this. Note that the eample below involves initial Dirac distributions, where there is a single relevant initial state only. Therefore, also in the special case of unique starting states transitivity for exact lumpability fails.

Example Consider the Markov chains M_1, M_2, M_3 depicted below.



Let $\ell: S_1 \to S_2$ and $k: S_2 \to S_3$ be such that $\ell(1) = a$, $\ell(2) = \ell(3) = b$, $\ell(4) = c$ and $k(a) = k(b) = x$, $k(c) = y$. The reader easily verifies that $M_1 \xrightarrow{\ell}_e M_2$ and $M_2 \xrightarrow{k}_e M_3$. However, M_1 admits no lumping to a two element chain such as M_3. The mapping $h: S_1 \to S_3$ with $h(1) = h(2) = h(3) = x$, $h(4) = y$ fails to satisfy condition III.4(ii).

V. Strict Confluence

Strict confluence can be interpreted as a reconciliation property: Suppose we have lumped a Markov chain M into a Markov chain M_1 via a lumping ℓ when focussing on one aspect A of a systems and that we have lumped M into another Markov chain M_2 via a lumping k for the analysis of some other aspect B. Is it possible to combine these intermediate chains to obtain quantitative information on A and B at the same time? The result we provide for ordinary lumping below constructs a Markov chain M' and lumpings h, f (determined by M_1, M_2 and the lumpings ℓ, k) such that M_1 and M_2 lump to M' via h and f, respectively.

Lemma V.1 *Suppose $M \xrightarrow{\ell}_o M_1$ and $M \xrightarrow{k}_o M_2$ for Markov chains M, M_1, M_2. Let $\sim$ be the equivalence relation on S induced by ℓ and k. Define the Markov chain $M' = (S', P', \pi')$ as follows:*

$$S' = S/\!\sim,\ P'([s], [t]) = \sum_{t'\sim t} P(s, t'),\ \pi'([s]) = \sum_{s'\sim s} \pi(s).$$

Then it holds that $M_1 \xrightarrow{h}_o M'$ and $M_2 \xrightarrow{f}_o M'$ where $h(u) = [s]$ for any $s \in \ell^{-1}(u)$ and $f(w) = [s]$ for any $s \in k^{-1}(w)$.

Proof As to the well-definedness of M', suppose $s \in S$. Then we have, for any $t \in S$,

$\sum_{t' \sim t} P(s, t')$
$= [[t] = \bigcup\{ \ell^{-1}(v) \mid \ell^{-1}(v) \subseteq [t] \}]$
$\sum_{\ell^{-1}(v) \subseteq [t]} \sum_{t' \in \ell^{-1}(v)} P(s, t')$
$=$ [ℓ is ordinary lumping] $\sum_{\ell^{-1}(v) \subseteq [t]} P_1(\ell(s), v)$.

Thus $\sum_{t' \sim t} P(s,' t) = \sum_{t' \sim t} P(s', t')$ for $s, s' \in S$ such that $\ell(s) = \ell(s')$. Likewise, using that k is an ordinary lumping, we get if $k(s) = k(s')$ then $\sum_{t' \sim t} P(s,' t) = \sum_{t' \sim t} P(s', t')$. From this we obtain that $s \sim s'$ implies $\sum_{t' \sim t} P(s,' t) = \sum_{t' \sim t} P(s', t')$ and that $P'([s], [t])$ is well-defined. (See Section II for the definition of $\sim$.)

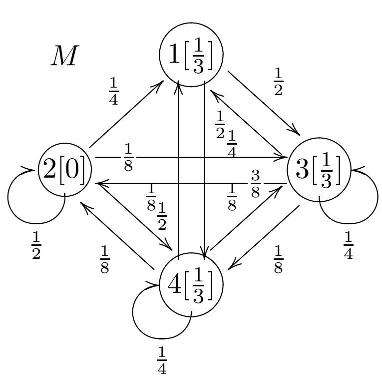
Clearly, $h\colon S_1 \to S'$ and $f\colon S_2 \to S'$ are well-defined and surjective. We have, by defintion of h, that $v \in h^{-1}([t]) \iff t \in \ell^{-1}(v) \iff \ell^{-1}(v) \subseteq [t]$ for $v \in S_1$, $t \in S$. Thus, for any $u \in S_1$, $s, t \in S$ such that $\ell(s) = u$ it holds that

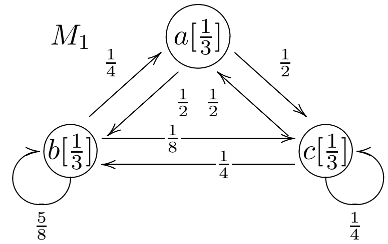
$P'(h(u), [t])$
$= P'([s], [t])$
$=$ [definition P'] $\sum_{t' \sim t} P(s, t')$
$=$ [decompostion $[t]$] $\sum_{\ell^{-1}(v) \subseteq [t]} \sum_{t' \in \ell^{-1}(v)} P(s, t')$
$= [M \xrightarrow{\ell}_o M_1]$ $\sum_{v \in h^{-1}([t])} P_1(u, v)$

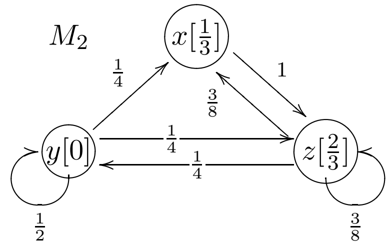
from which $M_1 \xrightarrow{h}_o M'$ follows. Similarly we obtain $M_2 \xrightarrow{f}_o M'$. ■

Every Markov chain can be generally lumped to the degenerated one element Markov chain, as can be readily checked from Definition III.1. However, this is not what one wants when analyzing concrete systems. Unfortunately, the notion of general lumpability does not allow for the construction used for ordinary lumpability in the Lemma V.1 above. The next counterexample illustrates this.

Example Consider the Markov chains M, M_1, M_2 given by



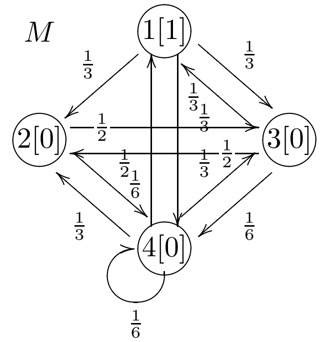


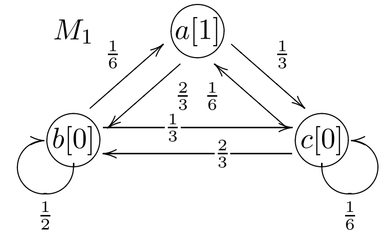


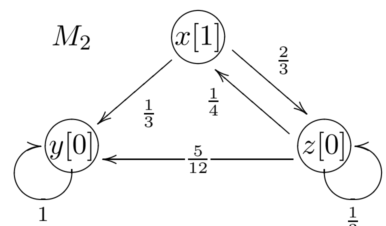
Here we have $\ell(1) = a, \ell(2) = \ell(3) = b, \ell(4) = c$ and $k(1) = x, k(2) = y, k(3) = k(4) = z$. With appeal of Lemma III.3 and Lemma III.5, it is easy to verify that $M \xrightarrow{\ell}_g M_1$ and $M \xrightarrow{k}_g M_2$, since $M \xrightarrow{\ell}_o M_1$ and $M \xrightarrow{\ell}_e M_2$. However, the construction in the proof of Lemma V.1 would violate condition III.1(ii) already for the case where $i = 2$ as the industrious reader may verify.

Next we show that the notion of exact lumpability is not strictly confluent at all.

Example Let the Markov chains M, M_1, M_2 be depicted as follows:







It holds that $M \xrightarrow{\ell}_e M_1$ and $M \xrightarrow{k}_e M_2$ with ℓ and k given by $\ell(1) = a, \ell(2) = \ell(3) = b, \ell(4) = c$ and $k(1) = x, k(2) = y, k(3) = k(4) = z$. However, M_1 and M_2 do not admit a common exact lumping. Because of condition III.3(i) on the initial probability distribution, for M_1 the states b and c should be lumped and for M_2 the states y and z, but they lead to different transition probabilities.

VI. Concluding remarks

In the above we have reviewed the concept and some properties of lumpability for discrete time Markov chains. Some of the results presented here are implicitly available in other work, see e.g. [Hil95], [Buc94b]. The present paper presents a self-contained and complete picture on transitivity and strict confluence, two properties relevant to tool based analysis of probabilistic systems.

It should be noted that the concepts discussed above and the results obtained, apply to continuous time Markov chains and Markov reward processes too. In the near future we plan to study the superposition of lumpability and the reduction techniques proposed by Voeten et al. reported in [PVT01], [BVP01].

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