Ansatz methods and resonance

For simplicity, all definitions and statements are for complex valued functions and vector spaces over $\mathbb{C}$.

For $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ we define the linear space of functions

$$V_{m,\lambda} := \{t \mapsto P_m(t)e^{\lambda t} \,|\, P_m \text{ polynomial of degree} \leq m\}.$$

The spaces $V_{m,\lambda}$ are interesting because they are invariant for linear differential operators with constant coefficients. Let L be such an operator, i.e. let L be defined by

$$L[y] := \sum_{k=0}^{n} a_k y^{(k)}, \qquad n \geq 1,\ a_n = 1,\ a_k \in \mathbb{C},$$

for all n times differentiable functions $y : \mathbb{R} \longrightarrow \mathbb{C}$. Let p be its characteristic polynomial, i.e.

$$p(\lambda) := \sum_{k=0}^{n} a_k \lambda^k.$$

Then, for any $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, the following holds:

(i) $L[V_{m,\lambda}] \subset V_{m,\lambda}$,

(ii) If $p(\lambda) \neq 0$ then $L : V_{m,\lambda} \longrightarrow V_{m,\lambda}$ is bijective.

(iii) If λ is a root of p with multiplicity $r > 0$, i.e.

$$p^{(k)}(\lambda) = 0 \text{ for all } k \leq r-1, \quad p^{(r)}(\lambda) \neq 0,$$

then $L : V_{m+r,\lambda} \longrightarrow V_{m,\lambda}$ is surjective.

Questions:

1. Can you see why these results ensure that the ansatz methods as discussed in the lecture always works?

2. ⋆ Try to prove them! (The proof is elementary but demands some "bookkeeping".)

The constant coefficent linear ODE

$$L[y] = f$$

is said to have resonance if $f \in V_{m,\lambda}$ and $p(\lambda) = 0$ for some $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. (More generally, the demand is that the right hand side can be split into linearly independent terms of which at least one lies in such a space $V_{m,\lambda}$.)