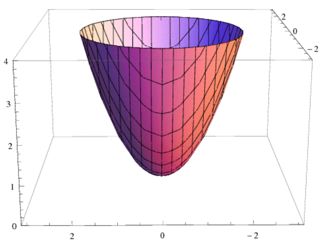
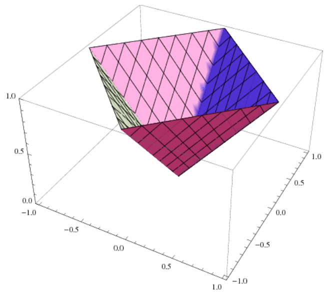
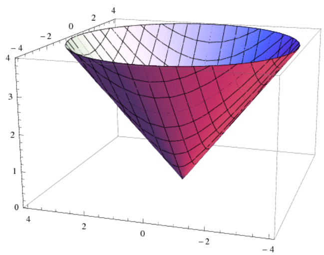
# 2 variables: graphs

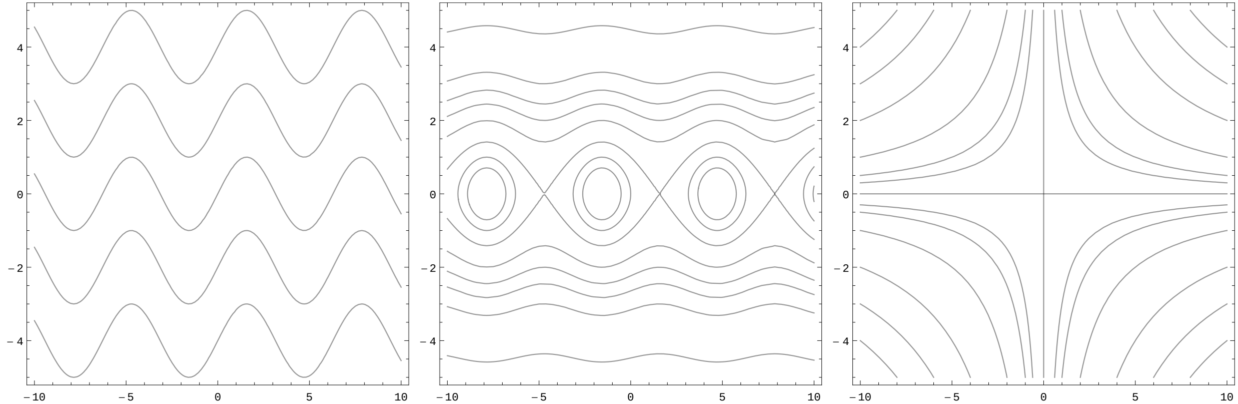


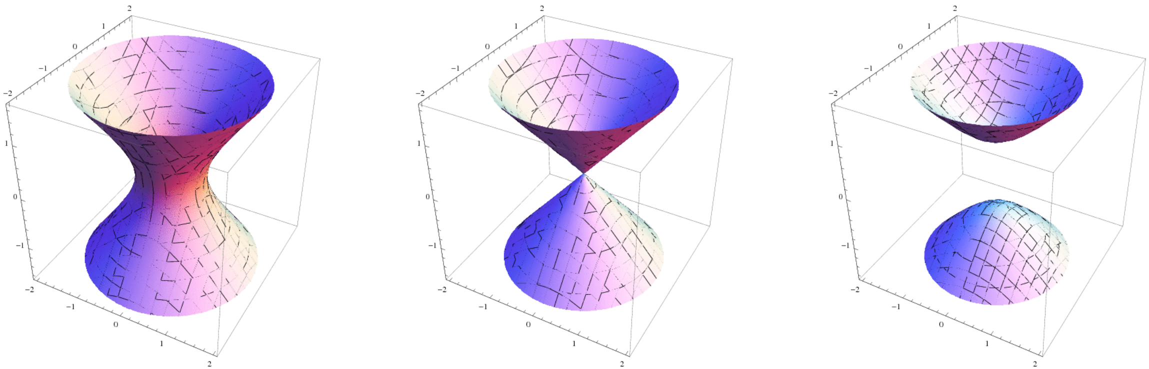

A: $f(x,y) = x^2 + y^2$
B: $f(x,y) = \sqrt{x^2 + y^2}$
C: $f(x,y) = |x| + |y|$

# 2 variables: level curves




**A:** $f(x, y) = \sin(x) + y^2$
**B:** $f(x, y) = y - \sin x$
**C:** $f(x, y) = xy$

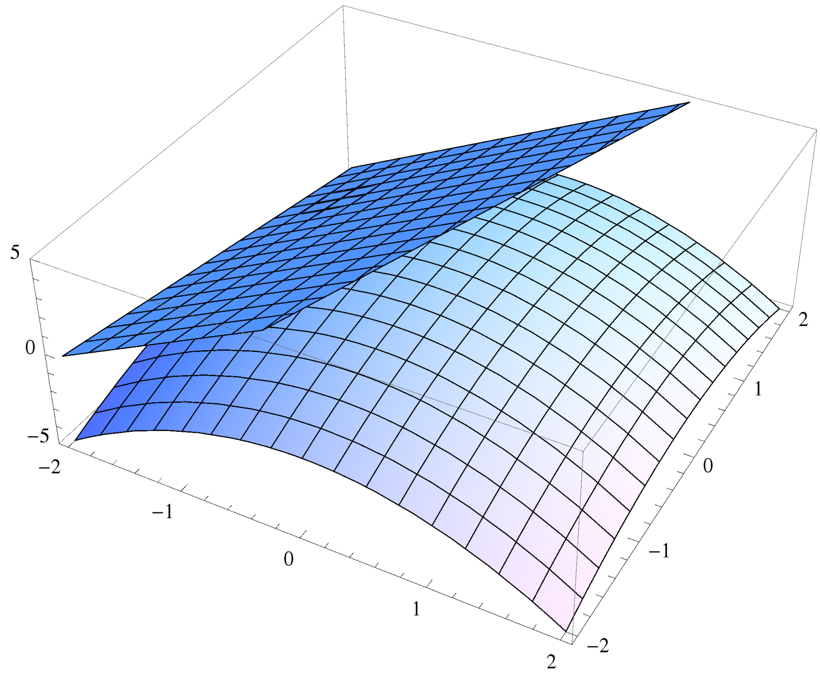
# 3 variables: level surfaces



**A:** $x^2 + y^2 - z^2 = 1/2$
**B:** $x^2 + y^2 - z^2 = -1/2$
**C:** $x^2 + y^2 - z^2 = 0$

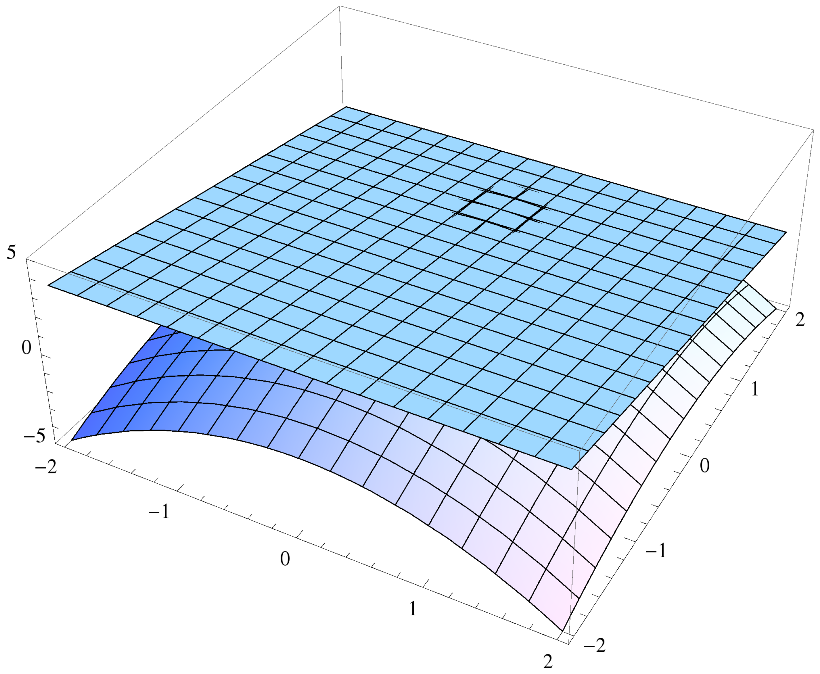
# Tangent planes to graphs

The graph of the linearization of $f$ is the tangent plane to the graph of $f$.



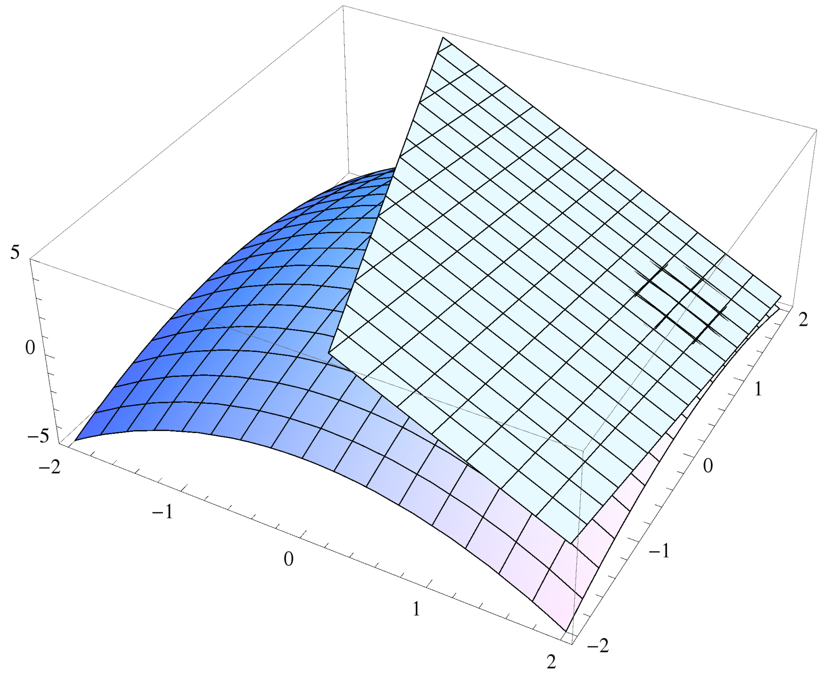
# Tangent planes to graphs

The graph of the linearization of $f$ is the tangent plane to the graph of $f$.



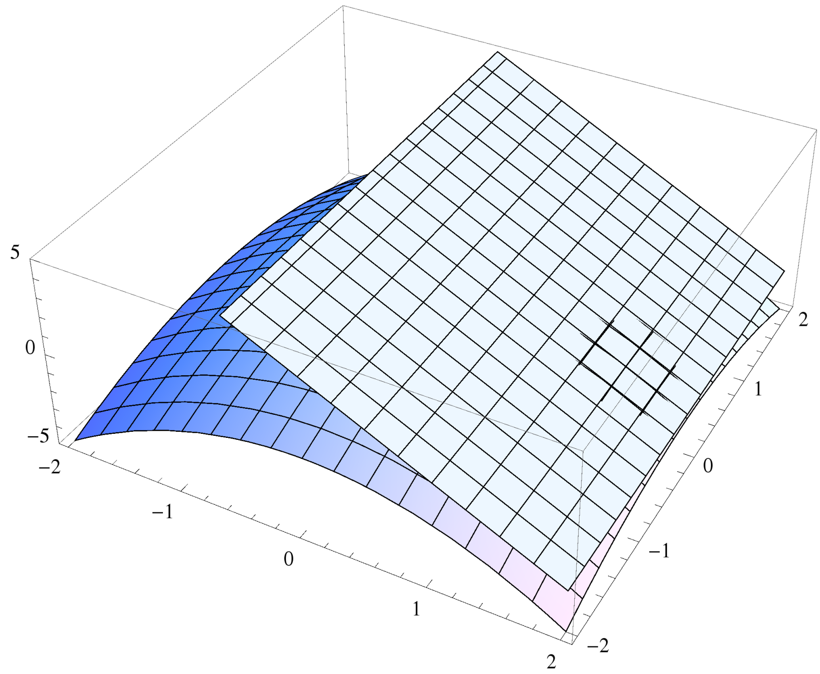
# Tangent planes to graphs

The graph of the linearization of $f$ is the tangent plane to the graph of $f$.



# Tangent planes to graphs

The graph of the linearization of $f$ is the tangent plane to the graph of $f$.



# Chain rule (2D)

Zij $g(t) = f(x(t), y(t))$, $f$, $x$, $y$ continuously differentiable. Then $g$ is differentiable, and

$$g'(t) = \partial_x f(x(t), y(t))x'(t) + \partial_y f(x(t), y(t))y'(t) = \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t).$$

Proof (sketch):

$$\frac{1}{h}[g(t+h) - g(t)]$$

# Chain rule (2D)

Zij $g(t) = f(x(t), y(t))$, $f$, $x$, $y$ continuously differentiable. Then $g$ is differentiable, and

$$g'(t) = \partial_x f(x(t), y(t))x'(t) + \partial_y f(x(t), y(t))y'(t) = \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t).$$

Proof (sketch):

$$\begin{aligned} & \tfrac{1}{h}[g(t+h) - g(t)] \\ = \quad & \tfrac{1}{h}[f(x(t+h), y(t+h)) - f(x(t), y(t))] \end{aligned}$$

# Chain rule (2D)

Zij $g(t) = f(x(t), y(t))$, $f$, $x$, $y$ continuously differentiable. Then $g$ is differentiable, and

$$g'(t) = \partial_x f(x(t), y(t))x'(t) + \partial_y f(x(t), y(t))y'(t) = \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t).$$

Proof (sketch):

$$\begin{aligned} & \tfrac{1}{h}[g(t+h) - g(t)] \\ = \quad & \tfrac{1}{h}[f(x(t+h), y(t+h)) - f(x(t), y(t))] \\ = \quad & \tfrac{1}{h}[f(x(t+h), y(t+h)) - f(x(t), y(t+h))] + \tfrac{1}{h}[f(x(t), y(t+h)) - f(x(t), y(t))] \end{aligned}$$

# Chain rule (2D)

Zij $g(t) = f(x(t), y(t))$, $f$, $x$, $y$ continuously differentiable. Then $g$ is differentiable, and

$$g'(t) = \partial_x f(x(t), y(t))x'(t) + \partial_y f(x(t), y(t))y'(t) = \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t).$$

Proof (sketch):

$$\begin{aligned}
& \tfrac{1}{h}[g(t+h) - g(t)] \\
= \quad & \tfrac{1}{h}[f(x(t+h), y(t+h)) - f(x(t), y(t))] \\
= \quad & \tfrac{1}{h}[f(x(t+h), y(t+h)) - f(x(t), y(t+h))] + \tfrac{1}{h}[f(x(t), y(t+h)) - f(x(t), y(t))] \\
\overset{h\to 0}{\to} \quad & \tfrac{d}{ds} f(x(s), y(t))|_{s=t} + \tfrac{d}{ds} f(x(t), y(s))|_{s=t}
\end{aligned}$$

# Chain rule (2D)

Zij $g(t) = f(x(t), y(t))$, $f$, $x$, $y$ continuously differentiable. Then $g$ is differentiable, and

$$g'(t) = \partial_x f(x(t), y(t))x'(t) + \partial_y f(x(t), y(t))y'(t) = \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t).$$

Proof (sketch):

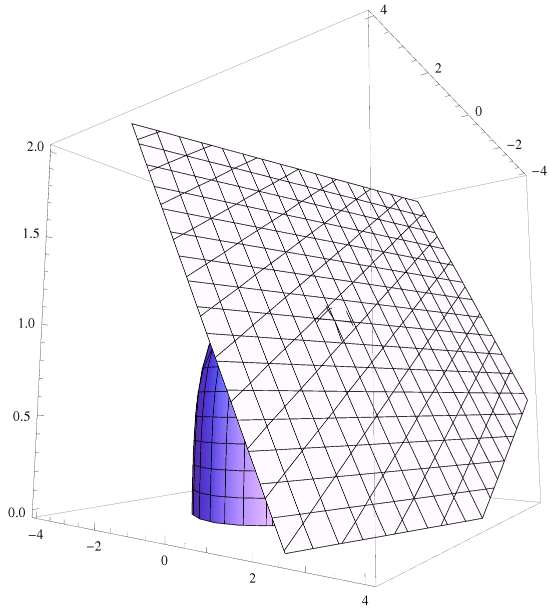
$$\begin{aligned}
& \tfrac{1}{h}[g(t+h) - g(t)] \\
= \quad & \tfrac{1}{h}[f(x(t+h), y(t+h)) - f(x(t), y(t))] \\
= \quad & \tfrac{1}{h}[f(x(t+h), y(t+h)) - f(x(t), y(t+h))] + \tfrac{1}{h}[f(x(t), y(t+h)) - f(x(t), y(t))] \\
\overset{h\to 0}{\to} \quad & \tfrac{d}{ds} f(x(s), y(t))|_{s=t} + \tfrac{d}{ds} f(x(t), y(s))|_{s=t} \\
\overset{\text{1D Chain rule}}{=} \quad & \partial_x f(x(t), y(t))x'(t) + \partial_y f(x(t), y(t))y'(t).
\end{aligned}$$

# Tangent planes to level surfaces

The tangent plane to the level surface $F(\mathbf{x}) = C$ in $\mathbf{x} = \mathbf{a}$ is given by

$$\nabla F(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0.$$

This is the corresponding level surface of the linearization of $F$ in $\mathbf{a}$.

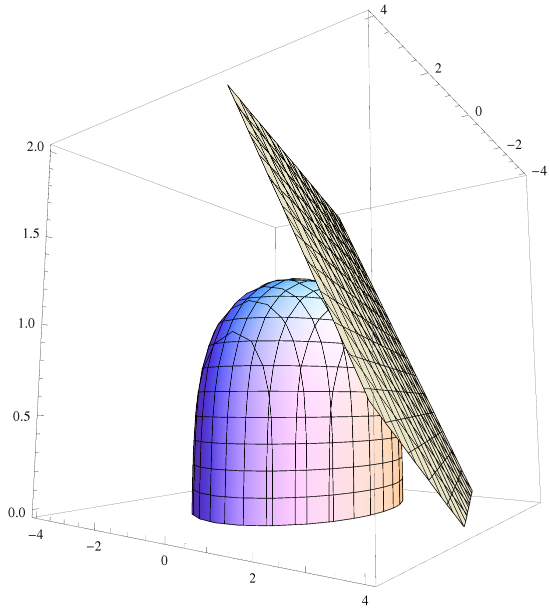


# Tangent planes to level surfaces

The tangent plane to the level surface $F(\mathbf{x}) = C$ in $\mathbf{x} = \mathbf{a}$ is given by

$$\nabla F(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0.$$

This is the corresponding level surface of the linearization of $F$ in $\mathbf{a}$.

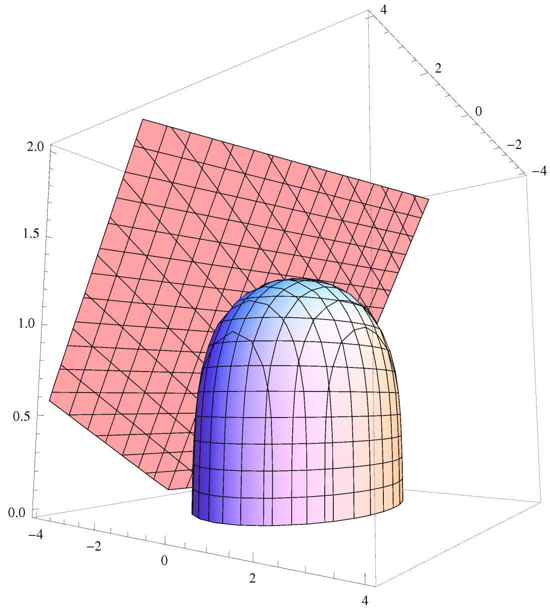


# Tangent planes to level surfaces

The tangent plane to the level surface $F(\mathbf{x}) = C$ in $\mathbf{x} = \mathbf{a}$ is given by

$$\nabla F(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0.$$

This is the corresponding level surface of the linearization of $F$ in $\mathbf{a}$.

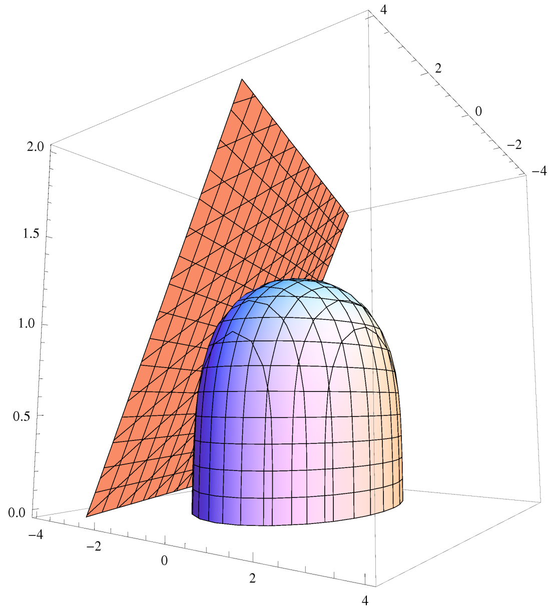


# Tangent planes to level surfaces

The tangent plane to the level surface $F(\mathbf{x}) = C$ in $\mathbf{x} = \mathbf{a}$ is given by

$$\nabla F(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0.$$

This is the corresponding level surface of the linearization of $F$ in $\mathbf{a}$.



## Differentiating an integral with respect to a parameter:

Let $J$ be an interval, $f : J \times [a,b] \longrightarrow \mathbb{R}$ such that:

- $f$ twice partially differentiable with respect to $x$
- $\int_a^b f(x,t)\,dt$, $\int_a^b f_x(x,t)\,dt$ exist for all $x \in J$
- $|f_{xx}(x,t)| \leq g(t)$, $\int_a^b g(t)\,dt = K < \infty$.

Then

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b f_x(x,t)\,dt, \qquad x \in J.$$

Proof (sketch): Write $\int_a^b f(x,t)\,dt =: I(x)$.

$$\left|\frac{1}{h}(I(x+h) - I(x)) - \int_a^b f_x(x,t)\,dt\right|$$

## Differentiating an integral with respect to a parameter:

Let $J$ be an interval, $f : J \times [a, b] \longrightarrow \mathbb{R}$ such that:

- $f$ twice partially differentiable with respect to $x$
- $\int_a^b f(x,t)\,dt$, $\int_a^b f_x(x,t)\,dt$ exist for all $x \in J$
- $|f_{xx}(x,t)| \leq g(t)$, $\int_a^b g(t)\,dt = K < \infty$.

Then

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b f_x(x,t)\,dt, \qquad x \in J.$$

Proof (sketch): Write $\int_a^b f(x,t)\,dt =: I(x)$.

$$\left|\frac{1}{h}(I(x+h) - I(x)) - \int_a^b f_x(x,t)\,dt\right| = \left|\frac{1}{h}\int_a^b (f(x+h,t) - f(x,t) - hf_x(x,t))\,dt\right|$$

## Differentiating an integral with respect to a parameter:

Let $J$ be an interval, $f : J \times [a,b] \longrightarrow \mathbb{R}$ such that:

- $f$ twice partially differentiable with respect to $x$
- $\int_a^b f(x,t)\,dt$, $\int_a^b f_x(x,t)\,dt$ exist for all $x \in J$
- $|f_{xx}(x,t)| \leq g(t)$, $\int_a^b g(t)\,dt = K < \infty$.

Then

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b f_x(x,t)\,dt, \qquad x \in J.$$

Proof (sketch): Write $\int_a^b f(x,t)\,dt =: I(x)$.

$$\left|\frac{1}{h}(I(x+h) - I(x)) - \int_a^b f_x(x,t)\,dt\right| = \left|\frac{1}{h}\int_a^b (f(x+h,t) - f(x,t) - hf_x(x,t))\,dt\right|$$

$$\overset{\text{Taylor}}{=} \left|\frac{1}{h}\int_a^b \frac{h^2}{2} f_{xx}(\xi,t)\,dt\right|$$

## Differentiating an integral with respect to a parameter:

Let $J$ be an interval, $f : J \times [a,b] \longrightarrow \mathbb{R}$ such that:

- $f$ twice partially differentiable with respect to $x$
- $\int_a^b f(x,t)\,dt$, $\int_a^b f_x(x,t)\,dt$ exist for all $x \in J$
- $|f_{xx}(x,t)| \le g(t)$, $\int_a^b g(t)\,dt = K < \infty$.

Then

$$\frac{d}{dx} \int_a^b f(x,t)\,dt = \int_a^b f_x(x,t)\,dt, \qquad x \in J.$$

Proof (sketch): Write $\int_a^b f(x,t)\,dt =: I(x)$.

$$\left| \frac{1}{h}(I(x+h) - I(x)) - \int_a^b f_x(x,t)\,dt \right| = \left| \frac{1}{h} \int_a^b (f(x+h,t) - f(x,t) - h f_x(x,t))\,dt \right|$$

$$\overset{\text{Taylor}}{=} \left| \frac{1}{h} \int_a^b \frac{h^2}{2} f_{xx}(\xi,t)\,dt \right| \le \frac{|h|}{2} \int_a^b |f_{xx}(\xi,t)|\,dt$$

## Differentiating an integral with respect to a parameter:

Let $J$ be an interval, $f : J \times [a,b] \longrightarrow \mathbb{R}$ such that:

- $f$ twice partially differentiable with respect to $x$
- $\int_a^b f(x,t)\,dt$, $\int_a^b f_x(x,t)\,dt$ exist for all $x \in J$
- $|f_{xx}(x,t)| \le g(t)$, $\int_a^b g(t)\,dt = K < \infty$.

Then

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b f_x(x,t)\,dt, \qquad x \in J.$$

Proof (sketch): Write $\int_a^b f(x,t)\,dt =: I(x)$.

$$\left|\frac{1}{h}(I(x+h)-I(x)) - \int_a^b f_x(x,t)\,dt\right| = \left|\frac{1}{h}\int_a^b (f(x+h,t)-f(x,t)-hf_x(x,t))\,dt\right|$$

$$\overset{\text{Taylor}}{=} \left|\frac{1}{h}\int_a^b \frac{h^2}{2} f_{xx}(\xi,t)\,dt\right| \le \frac{|h|}{2}\int_a^b |f_{xx}(\xi,t)|\,dt \le \frac{|h|}{2}\int_a^b g(t)\,dt$$

## Differentiating an integral with respect to a parameter:

Let $J$ be an interval, $f : J \times [a,b] \longrightarrow \mathbb{R}$ such that:

- $f$ twice partially differentiable with respect to $x$
- $\int_a^b f(x,t)\,dt$, $\int_a^b f_x(x,t)\,dt$ exist for all $x \in J$
- $|f_{xx}(x,t)| \le g(t)$, $\int_a^b g(t)\,dt = K < \infty$.

Then

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b f_x(x,t)\,dt, \qquad x \in J.$$

Proof (sketch): Write $\int_a^b f(x,t)\,dt =: I(x)$.

$$\left|\frac{1}{h}(I(x+h)-I(x)) - \int_a^b f_x(x,t)\,dt\right| = \left|\frac{1}{h}\int_a^b (f(x+h,t)-f(x,t)-hf_x(x,t))\,dt\right|$$

$$\overset{\text{Taylor}}{=} \left|\frac{1}{h}\int_a^b \frac{h^2}{2} f_{xx}(\xi,t)\,dt\right| \le \frac{|h|}{2}\int_a^b |f_{xx}(\xi,t)|\,dt \le \frac{|h|}{2}\int_a^b g(t)\,dt = K\frac{|h|}{2}$$

## Differentiating an integral with respect to a parameter:

Let $J$ be an interval, $f : J \times [a,b] \longrightarrow \mathbb{R}$ such that:

- $f$ twice partially differentiable with respect to $x$
- $\int_a^b f(x,t)\,dt$, $\int_a^b f_x(x,t)\,dt$ exist for all $x \in J$
- $|f_{xx}(x,t)| \le g(t)$, $\int_a^b g(t)\,dt = K < \infty$.

Then

$$\tfrac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b f_x(x,t)\,dt, \qquad x \in J.$$

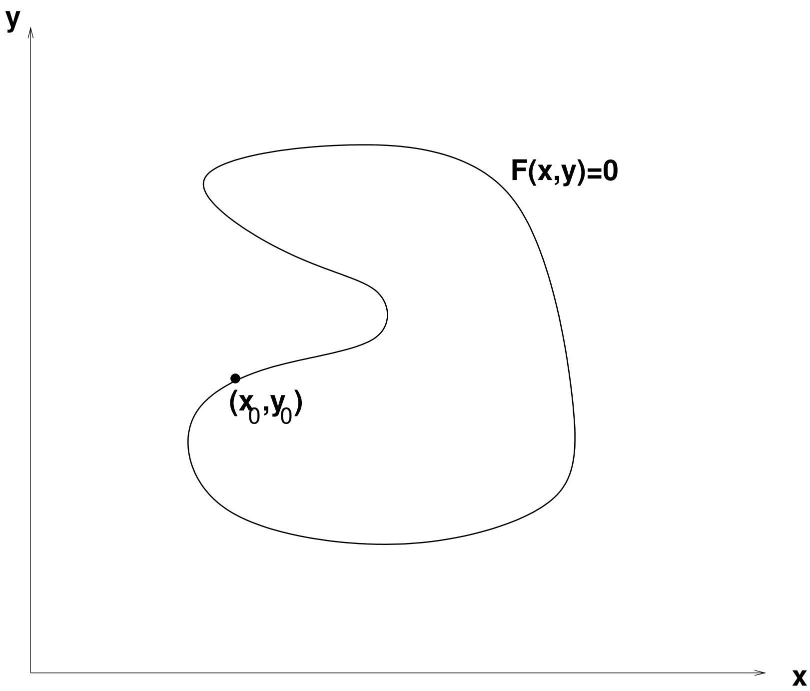
Proof (sketch): Write $\int_a^b f(x,t)\,dt =: I(x)$.

$$\left|\tfrac{1}{h}(I(x+h)-I(x)) - \int_a^b f_x(x,t)\,dt\right| = \left|\tfrac{1}{h}\int_a^b (f(x+h,t)-f(x,t)-hf_x(x,t))\,dt\right|$$

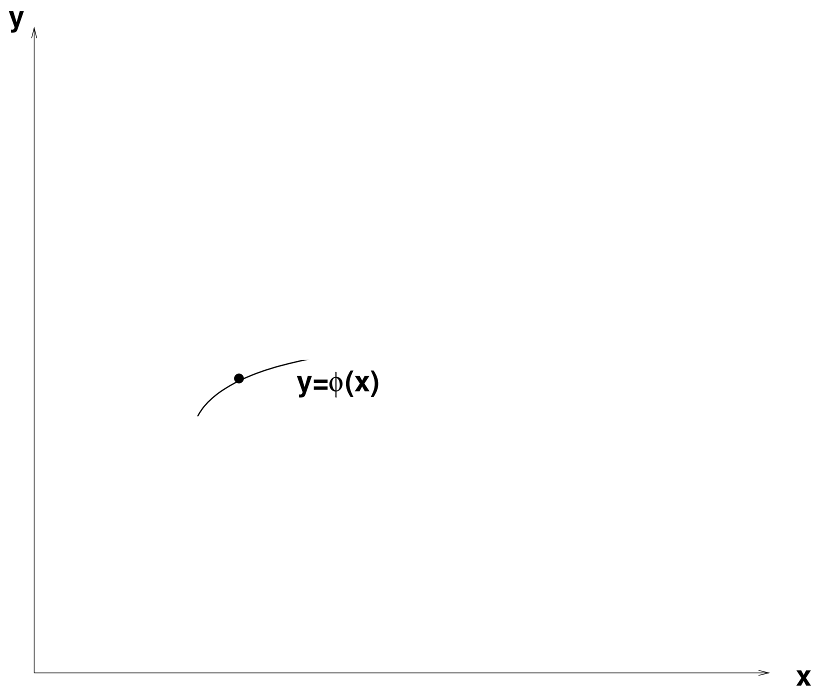
$$\overset{\text{Taylor}}{=} \left|\tfrac{1}{h}\int_a^b \tfrac{h^2}{2} f_{xx}(\xi,t)\,dt\right| \le \tfrac{|h|}{2}\int_a^b |f_{xx}(\xi,t)|\,dt \le \tfrac{|h|}{2}\int_a^b g(t)\,dt = K\tfrac{|h|}{2}$$

$$\overset{h\to 0}{\to} 0.$$

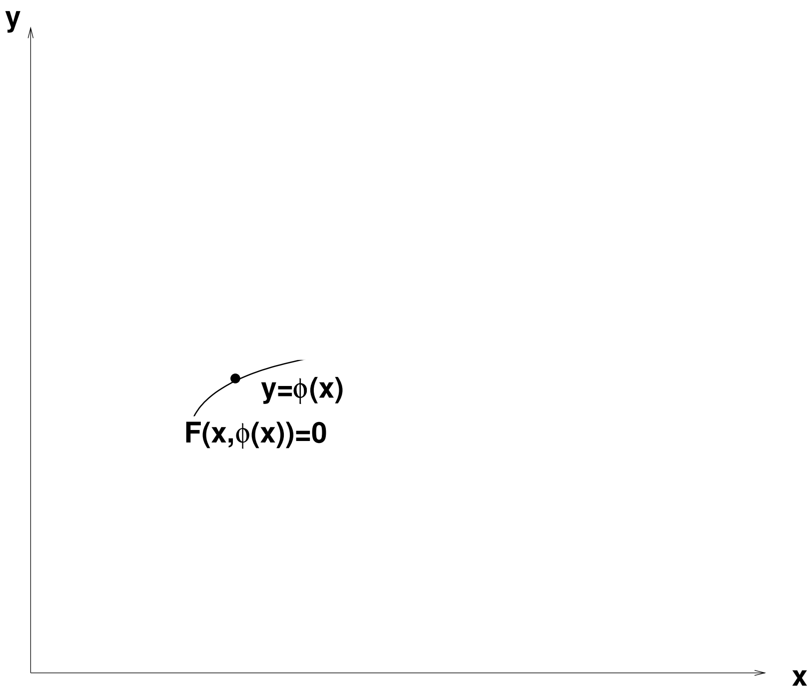
## Implicit Function Theorem (2 variables):



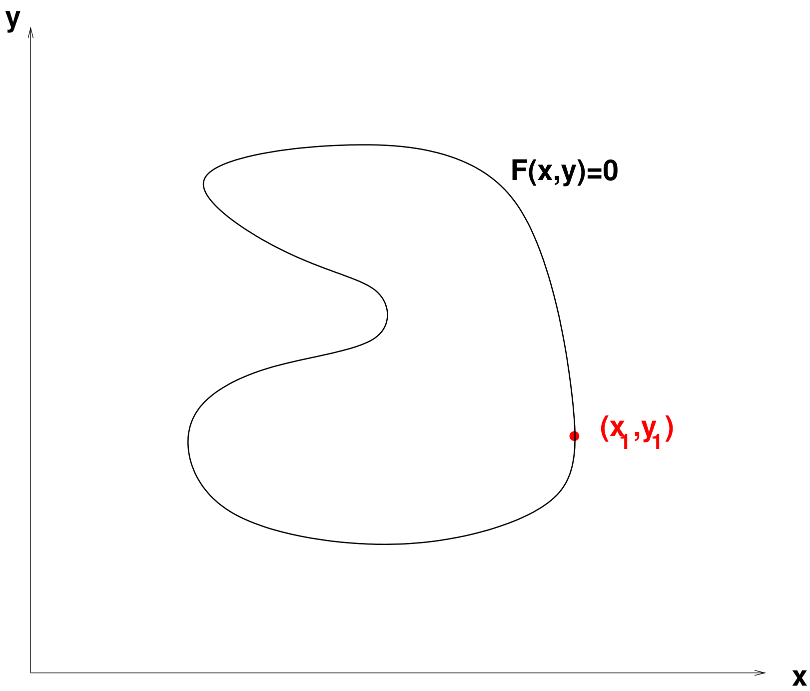
## Implicit Function Theorem (2 variables):



## Implicit Function Theorem (2 variables):



# Implicit Function Theorem (2 variables):



## Implicit Function Theorem (2 variables):