Homework Advanced Calculus (2DBN10)
Set 1

1. Let A be a real diagonalizable (n,n)-matrix. We consider the solutions $y : \mathbb{R} \longrightarrow \mathbb{R}^n$ of the linear homogeneous ODE system

$$\dot{y}(t) = Ay(t). \tag{1}$$

Give (necessary and sufficient) conditions in terms of the eigenvalues of A for the following statements to be true:

1 pt **a)** The system (1) has periodic solutions that are not identically equal to zero.

1 pt **b)** The system (1) has constant solutions that are not identically equal to zero.

1 pt **c)** For all solutions y of (1) there holds $\lim_{t\to\infty} y(t) = 0$.

1 pt **d)** For all solutions y of (1) there holds $\lim_{t\to-\infty} y(t) = 0$.

Give reasons for your answers.

2. This problem is aimed at understanding the structure of the solution space of second-order constant coefficients ODEs in the case of a double root of the characteristic polynomial ("resonance").

(**Warning to higher-year students:**
The problem is not identic to the similar one given last year!)

Fix $a \in \mathbb{R}$, let $\varepsilon > 0$ and consider the ODE

$$\ddot{y} - 2a\dot{y} + (a^2 - \varepsilon^2)y = 0. \tag{2}$$

2 pt **a)** Let $z_0, z_1 \in \mathbb{R}$. Solve the equation with initial conditions

$$y(0) = z_0, \quad \dot{y}(0) = z_1 \tag{3}$$

2 pt **b)** Denote the solution by y_ε. Calculate (for $t \in \mathbb{R}$)

$$y_0(t) := \lim_{\varepsilon\to 0} y_\varepsilon(t).$$

2 pt **c)** Show that y_0 satisfies

$$\ddot{y}_0 - 2a\dot{y}_0 + a^2 y_0 = 0 \tag{4}$$

with the initial conditions (3). Give the roots of the characteristic equation of (4). How are (2) and (4) related?

Exam type problem (not to be handed in): Consider the ODE system

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 2 \\ -7 & 4 & 4 \\ 6 & -4 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

a) Does this system have periodic solutions?

b) Find all solutions y that satisfy $|y(t)| \to 0$ as $t \to -\infty$.

Answers:

a) yes,

b) all solutions of the form

$$y(t) = C(1,1,1)^{\top} e^{t}, \quad C \in \mathbb{R}$$