Homework Advanced Calculus (2DBN10)
Set 2: The impulse response of a linear system

1. For $a > 0$ let $I_a : (0, \infty) \longrightarrow \mathbb{R}$ be given by

$$I_a(t) := \begin{cases} \dfrac{1}{a} & \text{if } t \leq a, \\ 0 & \text{if } t > a. \end{cases}$$

a) Sketch the graph of I_a for $a = 1,\ 0.5,\ 0.2.$ 1 pt

b) Calculate the Laplace transform $\mathcal{L}[I_a]$ and, for fixed $s > 0$, 2 pt

$$\lim_{a \downarrow 0} \mathcal{L}[I_a](s). \qquad (1)$$

c) Fix $\omega > 0$ and $a > 0$. Using Laplace transform, solve the initial value problem 3 pt

$$\ddot{y} + \omega^2 y = I_a, \qquad y(0) = \dot{y}(0) = 0.$$

Let y_a denote the solutions and find, for fixed $t > 0$, the limit

$$y_0(t) := \lim_{a \downarrow 0} y_a(t).$$

Calculate $\mathcal{L}[y_0]$, and compare the result to (1).

d) **(not to be handed in!)** Why is y_0 called the **impulse response** of the oscillator described by the equation $\ddot{y} + \omega^2 y = f$?

2. More generally, let $n \in \mathbb{N}$, $n \geq 1$, and let $a_0, \ldots a_{n-1} \in \mathbb{R}$ be fixed, $a_n = 1$. Let A be the linear differential operator given by

$$A[y] := \sum_{k=0}^{n} a_k y^{(k)} \qquad (y^{(0)} = y),$$

and p given by

$$p(\lambda) := \sum_{k=0}^{n} a_k \lambda^k$$

its characteristic polynomial.

Let U_0 be defined by

$$U_0(s) := \frac{1}{p(s)}.$$

The inverse Laplace transform u_0 of U_0 is called the **impulse response** of the system given by A.

Show that $\mathcal{L}[A[u_0]]$ is a polynomial in s. Use the fact that 4 pt

$$\lim_{s \to \infty} \mathcal{L}[A[u_0]](s) = 0$$

to conclude that $A[u_0] = 0$ and to find the initial values

$$u_0(0),\ \dot{u}_0(0),\ \dots\ u_0^{(n-1)}(0).$$

Hints: Do not try to calculate u_0 explicitly. You may assume without proof that u_0 is n times continuously differentiable on $[0, \infty)$, and that Au_0 is of exponential type. You can also use without proof that the Laplace transform is injective[1], i.e. if f is a continuous function of exponential type then

$$\mathcal{L}[f] \equiv 0 \quad \Longrightarrow \quad f \equiv 0.$$

Remark: You might have heard about distributions in general and the Delta distribution in particular. These are not covered in our course, and the problems (with the possible exception of 1d)) can and should be solved without using them in calculations or arguments.

Exam type problem (not to be handed in):

Find the solution to the ODE system

$$\begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}$$

with initial conditions

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(**Hint:** Using Laplace transform is the most efficient technique here.)

[1] In fact, we use this implicitly whenever we speak about "inverting the Laplace transform".