Homework Advanced Calculus (2DBN10)
Set 4

1. The ellipsoid E in $\mathbb{R}^3$ with equation 1 pt

$$2x^2 + y^2 + 3z^2 = 6$$

and the hyperboloid H with equation

$$2(x-2)^2 - y^2 + 3(z-3)^2 = 13$$

have a curve of intersection that contains the point $(1, 1, 1)$. Find the tangent line to the curve in that point.

(**Hint:** This tangent line lies in both the tangent planes to E and to H in the point $(1, 1, 1)$.)

2. (The Laplace operator and harmonic functions)

Let $D \subset \mathbb{R}^d$. The operator Δ acting on the twice continuously differentiable functions on D, given by

$$\Delta u := \sum_{i=1}^{d} \partial_{x_i}^2 u$$

is called **Laplace operator** or **Laplacian**. A twice continuously differentiable function $u : D \longrightarrow \mathbb{R}$ that satisfies $\Delta u = 0$ is called **harmonic** (in D). The Laplacian and harmonic functions play an important role e.g. in continuum mechanics and electrodynamics.

a) Find all harmonic polynomials in two variables (on $\mathbb{R}^2$) of degree 2. 1 pt

b) Verify that the function $u : \mathbb{R}^3 \setminus \{0\} \longrightarrow \mathbb{R}$ given by 1 pt

$$u(\underline{x}) = \frac{1}{|\underline{x}|}, \qquad \underline{x} \neq 0$$

is harmonic. (Give a clear account of your calculations.)

Are the following statements true? (If they are, explain clearly why. If they are not, show this by provinding a counterexample.)

c) "If u and v are harmonic functions on $\mathbb{R}^2$ then the pointwise product uv is harmonic as well." 1 pt

d) "If u is a harmonic function on $\mathbb{R}^2$, $\alpha > 0$ fixed, and v is given by 1 pt

$$v(x, y) = u(x\cos\alpha + y\sin\alpha, -x\sin\alpha + y\cos\alpha)$$

then v is harmonic as well."

e) "If u is a harmonic function on $\mathbb{R}^2$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f''(z) \geq 0$ for all $z \in \mathbb{R}$, and v is given by 1 pt

$$v(x, y) = f(u(x, y))$$

then $\Delta v \geq 0$ on $\mathbb{R}^2$.

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3. (The Laplacian in polar coordinates)

Let $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be twice continuously partially differentiable and let $\widetilde{u} : (0,\infty) \times (0, 2\pi) \longrightarrow \mathbb{R}$ be given by

$$\widetilde{u}(r,\theta) = u(r\cos\theta, r\sin\theta).$$

a) Show: 2 pt

$$\partial_1 u(r\cos\theta, r\sin\theta) = \cos\theta\, \partial_r \widetilde{u}(r,\theta) - \frac{\sin\theta}{r}\, \partial_\theta \widetilde{u}(r,\theta),$$
$$\partial_2 u(r\cos\theta, r\sin\theta) = \sin\theta\, \partial_r \widetilde{u}(r,\theta) + \frac{\cos\theta}{r}\, \partial_\theta \widetilde{u}(r,\theta).$$

(**Hint:** First, use the chain rule to represent $\partial_r \widetilde{u}(r,\theta)$ and $\partial_\theta \widetilde{u}(r,\theta)$ in terms of $\partial_1 u$ and $\partial_2 u$. Then, solve the linear system you got for $\partial_1 u$ and $\partial_2 u$.)

b) Give $\Delta u(r\cos\theta, r\sin\theta) = \partial_{11} u(r\cos\theta, r\sin\theta) + \partial_{22} u(r\cos\theta, r\sin\theta)$ in terms of partial derivatives of $\widetilde{u}$ with respect to r and θ. 2 pt

(**Hint:** Apply **a)** to $\partial_1 u$ and $\partial_2 u$ instead of u.)

(**Note:** Looking up the final result is easy and helpful as a check, but not the objective of this exercise. Make sure you understand why the result holds, and describe your calculations in sufficient detail to convince a reader of it.)