Homework Advanced Calculus (2DBN10)
Set 5

1. (Verifying the second derivative test in a simple case)
Let $a, b, c \in \mathbb{R}$ and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by

$$f(x, y) = \tfrac{1}{2}(ax^2 + 2bxy + cy^2).$$

Verify that $(0, 0)$ is a critical point for f and the Hessian there is

$$\nabla^2 f(0, 0) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

a) Suppose $a \neq 0$ and show: 3 pt

(i) If $ac - b^2 > 0$ and $a > 0$ then f has a strict local minimum at $(0, 0)$.
(ii) If $ac - b^2 > 0$ and $a < 0$ then f has a strict local maximum at $(0, 0)$.
(iii) If $ac - b^2 < 0$ then $(0, 0)$ is a saddle point for f.

(**Hint:** Obviously, you cannot use the second derivative test in this problem, as we want to verify it. Instead, substitute $x = r\cos\theta$, $y = r\sin\theta$ and check first that

$$f(x, y) = \tfrac{r^2}{2a}\left((a\cos\theta + b\sin\theta)^2 + (ac - b^2)\sin^2\theta\right).\quad)$$

b) Now suppose $a = 0$ and $ac - b^2 (= -b^2) < 0$. Show that then $(0, 0)$ is a saddle point. 2 pt

2. (Distance minimizing)

a) (A preliminary result from vector geometry) 2 pt
Let $v_1, v_2, v_3 \in \mathbb{R}^2 \setminus \{0\}$ be nonzero vectors in the plane. Show that if

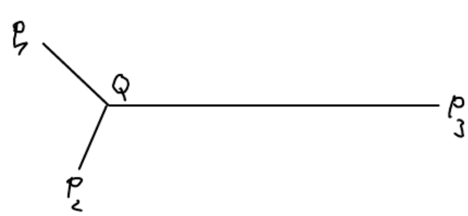
$$|v_1| = |v_2| = |v_3| \quad \text{and} \quad v_1 + v_2 + v_3 = 0$$

then any two of the vectors include an angle of $120°$.

b) Let P_1, P_2, P_3 be three points in the plane and let Q be a fourth point (different from all P_i) such that the total distance 3 pt

$$|\overline{P_1Q}| + |\overline{P_2Q}| + |\overline{P_3Q}| \tag{1}$$

is minimal (over all possible choices of Q, with P_i fixed).
Sketch:



Show that the angles between the lines $\overline{P_1Q}$, $\overline{P_2Q}$, $\overline{P_3Q}$ are all $120°$.
(**Remark:** Roughly speaking, this is the reason why area minimizing surfaces, as e.g. soap films, meet under angles of $120°$.)

c) $\star$ (not to be handed in): In b) we explicitly demanded that Q does not coincide with any of the P_i. (Where is this assumption used?)

However, for certain choices of P_1, P_2, P_3 the total distance (1) is minimized if Q does coincide with one of the points P_i. Under what conditions on the points P_i does this happen?

3. (Exam type question, not to be handed in)

Find the global maxima and minima of the function f given by

$$f(x, y) = 2x^3 + y^4$$

on the closed unit disk $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.