# Homework Advanced Calculus (2DBN10)
# Set 6

**1.** Let $D = \mathbb{R}^2 \setminus \{0\}$ and let $f : D \longrightarrow \mathbb{R}^2$ be the coordinate transformation given by 5 pt

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \operatorname{Re}\left(\frac{1}{u+iv}\right) \\ \operatorname{Im}\left(\frac{1}{u+iv}\right) \end{pmatrix} = \frac{1}{u^2+v^2} \begin{pmatrix} u \\ -v \end{pmatrix}.$$

Sketch some coordinate lines and describe them in geometric terms.

(**Hints:** To find the coordinate lines with $u = \text{const.}$, calculate

$$\left(\frac{u}{u^2+v^2} - \frac{1}{2u}\right)^2 + \left(\frac{v}{u^2+v^2}\right)^2 \qquad (u \neq 0)$$

and simplify the result as far as possible.

For $v = \text{const.}$, use a similar identity.

To get the correct idea, you may use graphing software.)

**2.** Let $S$ be a closed curve in $\mathbb{R}^2$, given by 5 pt

$$S = \{(x,y) \,|\, G(x,y) = 0\}$$

with some given differentiable function $G : \mathbb{R}^2 \longrightarrow \mathbb{R}$ that satisfies $\nabla G \neq 0$ on $S$. Let $P_i \in \mathbb{R}^2$ a given point inside $S$ and $P_e \in \mathbb{R}^2$ a given point outside $S$. Let $a_i, a_e > 0$ be given. Let $Q \in S$ be such that the sum of distances

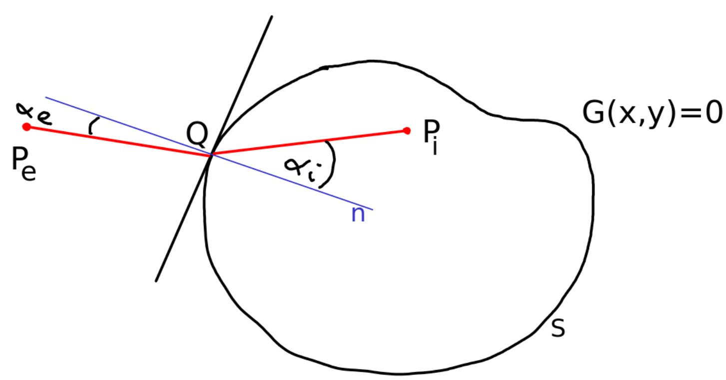
$$a_i|\overline{P_iQ}| + a_e|\overline{P_eQ}|$$

is minimal (among all possible choices of $Q$, $P_i$, $P_e$ fixed.

Let $n$ be the line normal to the tangent plane to $S$ in $Q$.

Let $\alpha_{i,e} := \measuredangle(n, \overline{P_{i,e}Q})$.

Sketch:

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We assume that $P_e$ and $P_i$ are not on $n$. Verify that

$$\frac{\sin \alpha_i}{\sin \alpha_e} = \frac{a_e}{a_i}.$$

(**Remark:** In a context of geometric optics, this is a derivation of Snell's law of refraction of light rays from Fermat's principle of minimal time.)

**3.** (Exam type question, not to be delivered)

Consider the system of equations

$$\begin{aligned} x^3 + y^4 + u^3 + v^2 &= 4 \\ x^3 + y^7 - y + 2u + v^4 &= 4. \end{aligned}$$

Show: In a neighborhood of the point

$$(x_0, y_0, u_0, v_0) = (1, 1, 1, 1)$$

this system is solvable for $x = \phi(u, v)$, $y = \psi(u, v)$. Calculate the partial derivatives $\partial_u \phi$, $\partial_u \psi$, $\partial_v \phi$, $\partial_v \psi$ in the point $(u, v) = (1, 1)$.