Laplace transform

Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be piecewise continuous and assume that there are M, $s_0 \geq 0$ such that

$$|f(t)| \leq M e^{s_0 t}, \qquad t > 0.$$

Then the function $F : (s_0, \infty) \longrightarrow \mathbb{R}$ given by

$$\boxed{F(s) := \int_0^\infty e^{-st} f(t)\, dt, \qquad s > s_0}$$

is well-defined (check!). It is called the **Laplace transform** of f.

Notations: $F = \mathcal{L}[f]$, $F = \hat{f}$, $F = \tilde{f}$, and others.

Basic properties: (Check!)

Linearity: If $f, g : (0, \infty) \longrightarrow \mathbb{R}$ both have Laplace transforms, then $f + g$ also has a Laplace transform, and

$$\mathcal{L}[f + g] = \mathcal{L}[f] + \mathcal{L}[g]$$

on the interval where both are defined. If $c \in \mathbb{R}$ then cf also has a Laplace transform, and

$$\mathcal{L}[cf] = c\mathcal{L}[f].$$

Shifting: • If f has a Laplace transform F on (s_0, ∞) and $a \in \mathbb{R}$ then the function g given by

$$g(t) = e^{at} f(t)$$

has a Laplace transform G on $(\max(s_0 + a, 0), \infty)$, and

$$G(s) = F(s - a)$$

on this interval.

• Let $a > 0$. If f has a Laplace transform F on (s_0, ∞) then the function g given by

$$g(t) = \begin{cases} f(t - a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$$

has a Laplace transform G on (s_0, ∞), and

$$G(s) = e^{-as} F(s)$$

on this interval.

Scaling: Let $a > 0$. If f has a Laplace transform F on (s_0, ∞) then the function g given by

$$g(t) = f(at)$$

has a Laplace transform G on (as_0, ∞), and

$$G(s) = \tfrac{1}{a} F\left(\tfrac{s}{a}\right)$$

on this interval.

Derivatives: If f has a derivative g having a Laplace transform G on the interval (s_0, ∞) then f has a Laplace transform on the same interval, and

$$G(s) = sF(s) - f(0).$$

More generally, for higher derivatives we have (under analogous assumptions)

$$\mathcal{L}[f^{(n)}](s) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0).$$