A general solution method for homogeneous linear ODE systems with constant coefficients

Let A be a (not necessarily diagonalizable, real or complex) (n, n)-matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_r$ with corresponding algebraic multiplicities $m_1, \dots, m_r$.

To determine the general solution of the ODE system

$$\dot{y} = Ay \tag{1}$$

we use the ansatz

$$y(t) = \sum_{j=1}^{r} e^{\lambda_j t} \sum_{k=0}^{m_j-1} v_{j,k} \frac{t^k}{k!}, \quad v_{j,k} \in \mathbb{C}^n,$$

with the n vectors $v_{j,k}$ to be determined.

Inserting this ansatz into (1) yields homogeneous linear systems

$$(A - \lambda_j I) v_{j,k} = \begin{cases} v_{j,k+1} & \text{if } k < m_j - 1, \\ 0 & \text{if } k = m_j - 1, \end{cases} \qquad j = 1, \dots, r,$$

for the ansatz vectors $v_{j,0}, \dots, v_{j,m_j-1}$. (Check!)

It can be shown that these systems (of dimension $nm_j \times nm_j$) have solution spaces of dimension m_j. If bases and and free parameters are chosen accordingly, the ansatz yields a linear combination of $m_1 + \dots + m_r = n$ independent solutions to (1), i.e. the general solution.