Systems of linear ODEs with constant coefficients

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $n \in \mathbb{N}$, $A \in \mathbb{R}^{n\times n}$.

Seek differentiable functions $y : \mathbb{R} \longrightarrow \mathbb{K}^n$ such that

$$\dot{y}(t) = Ay(t), \qquad t \in \mathbb{R}.$$

Remark: The solutions form a linear space. (Check!) It has dimension n.

Therefore, the general solution can be written as

$$y(t) = \sum_{k=1}^{n} c_k y_k(t), \quad c_k \in \mathbb{K},$$

where $\{y_1, \ldots, y_n\}$ is a linearly independent set of solutions (i.e. a basis of the solution space.)

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Assume now A diagonalizable, and let $\{v_1, \dots, v_n\}$ be a basis of $\mathbb{K}^n$ consisting of eigenvectors of A, i.e. $Av_k = \lambda_k v_k$, $k = 1, \dots, n$, or

$$A \underbrace{\begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}}_{V} = \underbrace{\begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}}_{D},$$

i.e. $A = VDV^{-1}$, and with $z(t) = V^{-1}y(t)$

$$\dot{z} = V^{-1}\dot{y} = V^{-1}VDV^{-1}y = Dz,$$

i.e. $\dot{z}_k = \lambda_k z_k$, so $z_k = c_k e^{\lambda_k t}$, and

$$y(t) = Vz(t) = \sum_{k=1}^{n} c_k \underbrace{v_k e^{\lambda_k t}}_{\text{“modes”}}$$

(general solution).

Systems of linear ODEs with constant coefficients

Complex eigenvalues: Let $\lambda_k \notin \mathbb{R}$ be a complex eigenvalue of A with eigenvector v_k. (Then $v_k \notin \mathbb{R}^n$!)

Then $\bar{\lambda}_k \neq \lambda_k$ is also an eigenvalue of A, and $\bar{v}_k$ is a corresponding eigenvector. (Check!)

So $v_k e^{\lambda_k t}$ and $\bar{v}_k e^{\bar{\lambda}_k t} = \overline{v_k e^{\lambda_k t}}$ are two linearly independent elements of the solution space.

Instead of these, we can also choose the real solutions

$$\begin{aligned} \frac{1}{2}(v_k e^{\lambda_k t} + \overline{v_k e^{\lambda_k t}}) &= \mathrm{Re}(v_k e^{\lambda_k t}), \\ \frac{1}{2i}(v_k e^{\lambda_k t} - \overline{v_k e^{\lambda_k t}}) &= \mathrm{Im}(v_k e^{\lambda_k t}). \end{aligned}$$

Systems of linear ODEs with constant coefficients

Initial value problems: Let further $y_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$ be given.

Find a function $y : \mathbb{R} \longrightarrow \mathbb{R}^n$ such that

$$\dot{y} = Ay, \quad y(t_0) = y^0.$$

Solution: Determine the free constants c_k in the general solution such that

$$\sum_{k=1}^{n} c_k y_k(t_0) = y^0.$$

Systems of linear ODEs with constant coefficients

Inhomogeneous systems: Let $I \subset \mathbb{R}$ be an interval, $f : I \longrightarrow \mathbb{R}^n$ continuous.

Find functions $y : I \longrightarrow \mathbb{R}^n$ such that

$$\dot{y}(t) = Ay(t) + f(t), \quad t \in I. \qquad \text{(i)}$$

Theorem: Let $y_p : I \longrightarrow \mathbb{R}^n$ a ("particular") solution for (i) and y_h the general solution of the homogeneous system $\dot{y}_h = Ay_h$. Then the general solution of the inhomogeneous system (i) is given by

$$y(t) = y_p(t) + y_h(t), \qquad t \in I.$$

(Check!)

Systems of linear ODEs with constant coefficients

Finding a particular solution:

"Variation of parameters":

Let $\{y_1, \ldots, y_n\}$ be a basis for the solution space of the homogeneous system.

Ansatz:

$$y_p(t) = \sum_{k=1}^{n} c_k(t) y_k(t) = \underbrace{\begin{pmatrix} | & & | \\ y_1(t) & \ldots & y_n(t) \\ | & & | \end{pmatrix}}_{Y(t)} \underbrace{\begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix}}_{c(t)},$$

where $c_1(t), \ldots c_n(t) : I \longrightarrow \mathbb{R}$ are to be determined.

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$$\Rightarrow \qquad \sum_{k=1}^{n} \dot{c}_k(t) y_k(t) = Y(t)\dot{c}(t) = f(t)$$

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$$\Rightarrow \qquad \dot{c}(t) = Y(t)^{-1} f(t)$$

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\Rightarrow \quad & \sum_{k=1}^{n} \dot{c}_k(t) y_k(t) = Y(t)\dot{c}(t) = f(t) \\
\Rightarrow \quad & \dot{c}(t) = Y(t)^{-1} f(t) \\
\Rightarrow \quad & c(t) = \int Y(\tau)^{-1} f(\tau)\, d\tau + d, \quad d \in \mathbb{R}^n.
\end{aligned}$$

(vector valued integral)

Higher order ODEs and systems

Single equation:

$$u^{(n)} + a_{n-1}u^{(n-1)} + \ldots + a_1\dot{u} + a_0 u = f$$

Trick:

$$y_1 := u, \quad y_2 := \dot{u}, \quad \ldots \quad y_n := u^{(n-1)}.$$

$\Rightarrow$ **first-order system:**

$$\dot{y} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_n \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -a_0 & -a_1 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix}}_{A} y + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix}$$

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Observe: A has characteristic polynomial

$$P(\lambda) := |\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0.$$

Higher order ODEs and systems

General solution of the homogenous equation:

$\Rightarrow$ If P has n different roots $\lambda_k \in \mathbb{C}$ then

$$u_h(t) = \sum_{k=1}^{n} c_k e^{\lambda_k t}.$$

(The eigenvalues of A do not need to be calculated for this.
P can be directly read off from the equation!)

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The method of rewriting as first-order system also works for systems of higher-order ODEs.