Solutions to exercises Advanced Calculus (2DBN10) Lecture 3

(These solutions are deliberately sketchy in part. No rights can be derived from them.)

1. **a)** Solution to homogeneous system:

$$y(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}, \quad C_{1,2} \in \mathbb{R}.$$

Ansatz variation of parameters:

$$y(t) = C_1(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + C_2(t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}.$$

This gives

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-2t}C_1'(t) \\ e^{-4t}C_2'(t) \end{pmatrix} = \begin{pmatrix} 3\cos t \\ -2\cos t - 3\sin t \end{pmatrix}$$

with solution

$$\begin{aligned} C_1'(t) &= \tfrac{1}{2}e^{2t}(\cos t - 3\sin t), \\ C_2'(t) &= \tfrac{1}{2}e^{4t}(5\cos t + 3\sin t) \end{aligned}$$

and after integration

$$\begin{aligned} C_1(t) &= \tfrac{1}{2}e^{2t}(\cos t - \sin t) + D_1 \\ C_2(t) &= \tfrac{1}{2}e^{4t}(\cos t + \sin t) + D_2. \end{aligned}$$

So the general solution is

$$y(t) = D_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + D_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t} + \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \quad D_{1,2} \in \mathbb{R}.$$

b) Solution to homogeneous system:

$$y(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 4 \\ -5 \end{pmatrix} e^{2t}, \quad C_{1,2} \in \mathbb{R}.$$

Ansatz variation of parameters:

$$y(t) = C_1(t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + C_2(t) \begin{pmatrix} 4 \\ -5 \end{pmatrix} e^{2t}.$$

This gives

$$\begin{pmatrix} 1 & 4 \\ -1 & -5 \end{pmatrix} \begin{pmatrix} e^{t}C_1'(t) \\ e^{2t}C_2'(t) \end{pmatrix} = \begin{pmatrix} 5e^{t} \\ -6e^{t} \end{pmatrix}$$

with solution

$$C_1'(t) = 1, \quad C_2'(t) = e^{-t}$$

and after integration

$$C_1(t) = t + D_1, \quad C_2(t) = -e^{-t} + D_2.$$

So the general solution is

$$y(t) = D_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + D_2 \begin{pmatrix} 4 \\ -5 \end{pmatrix} e^{2t} + \begin{pmatrix} (t-4)e^t \\ (-t+5)e^t \end{pmatrix}, \quad D_{1,2} \in \mathbb{R}.$$

2. **a)** Characteristic polynomial

$$p(\lambda) = \lambda^3 - \lambda^2 + 9\lambda - 9$$

with zeros $\lambda_1 = 1$, $\lambda_{2,3} = \pm 3i$. So general solution

$$y(t) = C_1 e^t + C_2 \cos(3t) + C_3 \sin(3t).$$

b) Characteristic polynomial

$$p(\lambda) = \lambda^4 + 3\lambda^2 - 4$$

is biquadratic (so substitute $\mu = \lambda^2$). Zeros $\lambda_{1,2} = \pm 1$, $\lambda_{3,4} = \pm 2i$. So general solution

$$y(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos(2t) + C_4 \sin(2t).$$

3. **a)** The inhomogeneity e^{-t} is no solution of the homogeneous equation. So ansatz $y_p(t) = Ae^{-t}$ Filling this in gives $A = -1/20$.

b) The inhomogeneity $\sin t$ is no solution of the homogeneous equation. So ansatz $y_p(t) = A \cos t + B \sin t$. Filling this in gives $A = 0$, $B = -1/6$.