Exercises Advanced Calculus (2DBN10)
Lecture 4

1. Which of the following functions of the variable $t \in (0, \infty)$ have a Laplace transform? Why? (You are not expected to calculate it.)

$$\textbf{a)}\ t^t, \qquad \textbf{b)}\ 2^{(t^2)}, \qquad \textbf{c)}\ -e^{-t^3}.$$

2. Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be a bounded, piecewise continuous function with Laplace transform F. Let $a > 0$ be a fixed real number, and let $g : [0, \infty) \longrightarrow \mathbb{R}$ given by

$$g(t) = f(at).$$

Verify that the Laplace transform G of g is given by

$$G(s) = \tfrac{1}{a} F\left(\tfrac{s}{a}\right).$$

3. Use partial fraction decomposition, rules of calculation, and the table given on the course page to find the functions $t \mapsto f(t)$ whose Laplace transforms $s \mapsto \mathcal{L}[f](s)$ are given by

$$\textbf{a)}\ \frac{s-9}{s^2-9}, \quad \textbf{b)}\ \frac{1}{s^2(s+1)}, \quad \textbf{c)}\ \frac{s^3+6s^2+14s}{(s+2)^4}, \quad \textbf{d)}\ \frac{8}{s^4+4s^2}, \quad \textbf{e)}\ \frac{e^{-s}}{s^4}, \quad \textbf{f)}\ \frac{1-e^{-\pi s}}{s^2+4}.$$

4. Use Laplace transform to solve the following initial value problems:

a)

$$y'' - 5y' + 6y = r(t), \qquad y(0) = 1, \quad y'(0) = -2,$$

with r given by

$$r(t) = \begin{cases} 4e^t & 0 \le t \le 2, \\ 0 & t > 2. \end{cases}$$

b)

$$y'' + 9y = r(t), \qquad y(0) = 0, \quad y'(0) = 4,$$

with r given by

$$r(t) = \begin{cases} 8\sin t & 0 \le t \le \pi, \\ 0 & t > \pi. \end{cases}$$