

## Standard series expansions

All series given here are Taylor series around  $x_0 = 0$ .

### Geometric and related series

$$(1 - x)^{-1} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots$$

From this we find directly:

$$\begin{aligned}(1 \mp x^m)^{-1} &= \sum_{k=0}^{\infty} (\pm 1)^k x^{mk} = 1 \pm x^m + x^{2m} \pm \dots, \\ \ln(1 - x) &= -\int_0^x \frac{1}{1-t} dt = -\sum_{k=1}^{\infty} \frac{x^k}{k} = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots\right), \\ \arctan x &= \int_0^x \frac{1}{1+t^2} dt = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} \pm \dots\end{aligned}$$

The radius of convergence of these series is 1.

### Exponential and related series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

From this we find directly:

Even / odd terms:

$$\begin{aligned}\cosh x &= \frac{e^x + e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \dots, \\ \sinh x &= \frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{6} + \frac{x^5}{120} + \dots\end{aligned}$$

Real / Imaginary part of  $e^{ix}$ :

$$\begin{aligned}\cos x &= \operatorname{Re}(e^{ix}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} \pm \dots, \\ \sin x &= \operatorname{Im}(e^{ix}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} \pm \dots\end{aligned}$$

These series converge for all  $x \in \mathbb{R}$ .

## Binomial series

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1 \cdot 2} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{1 \cdot 2 \cdot 3} x^3 + \dots$$

where

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}.$$

The radius of convergence of this series is 1.

For  $\alpha \in \mathbb{N}_+$ , this series is a finite sum, found also by the binomial theorem.

For  $\alpha = -1$  we get the geometric series.

## Addition theorems for trigonometric functions

For  $\alpha, \beta \in \mathbb{R}$  we have

$$e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta},$$

so

$$\cos(\alpha + \beta) + i \sin(\alpha + \beta) = (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta).$$

Expanding and splitting in real and imaginary part yields

$$\begin{aligned} \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta, \\ \sin(\alpha + \beta) &= \cos \alpha \sin \beta + \sin \alpha \cos \beta. \end{aligned}$$