Multiindex notation

Multiindex:

$$\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d, \quad \alpha_i \in \mathbb{N}, \quad |\alpha| = \alpha_1 + \ldots + \alpha_d.$$

(Products of) powers:

$$\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d : \qquad \mathbf{x}^\alpha := x_1^{\alpha_1} \ldots x_d^{\alpha_d}.$$

Polynoms of degree $\leq n$:

$$p(\mathbf{x}) = \sum_{|\alpha| \leq n} c_\alpha \mathbf{x}^\alpha = \sum_{|\alpha| \leq n} \tilde{c}_\alpha (\mathbf{x} - \mathbf{x}_0)^\alpha.$$

Derivatives:

$$\partial^\alpha f = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d} f$$

for f sufficiently smooth.

(Theorem: Order of partial derivatives is interchangeable in that case.)

Calculation rules

Define for $\alpha, \beta \in \mathbb{N}^d$:

$$\alpha! = \alpha_1! \ldots \alpha_d! \quad \beta \leq \alpha \;\Leftrightarrow\; \beta_i \leq \alpha_i, i = 1, \ldots, d, \quad \alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_d - \beta_d) \text{ als } \beta \leq \alpha.$$

Then (check!):

- Binomial theorem:

$$(\mathbf{x} + \mathbf{y})^\alpha = \sum_{\beta \leq \alpha} \underbrace{\frac{\alpha!}{\beta!(\alpha - \beta)!}}_{\binom{\alpha}{\beta}} \mathbf{x}^\beta \mathbf{y}^{\alpha - \beta}, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d,\ \alpha \in \mathbb{N}^d.$$

- Derivatives of powers:

$$\partial^\beta \mathbf{x}^\alpha = \frac{\alpha!}{(\alpha - \beta)!} \mathbf{x}^{\alpha - \beta}, \qquad \beta \leq \alpha,\ \alpha, \beta \in \mathbb{N}^d.$$

- Higher order directional derivatives:

$$\partial_{\mathbf{v}}^k f = \sum_{|\alpha| = k} \frac{k!}{\alpha!} \mathbf{v}^\alpha \partial^\alpha f.$$

Taylor polynomial

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Find polynomial $T : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that

$$\partial^\beta T(\mathbf{a}) = \partial^\beta f(\mathbf{a}), \quad 0 \leq |\beta| \leq n.$$

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Choose $\mathbf{x} = \mathbf{a}$:

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So

$$\boxed{T(\mathbf{x}) = \sum_{|\alpha| \leq n} \frac{\partial^\alpha f(\mathbf{a})}{\alpha!} (\mathbf{x} - \mathbf{a})^\alpha}$$

(compare to 1D case!)

Taylor’s theorem

How large is the approximation error?

$$f(\mathbf{x}) = T_{n,a}(\mathbf{x}) + \ldots ?$$

Theorem (Taylor): Suppose $\mathbf{x} \in D$, and the line segment $[\mathbf{a}, \mathbf{x}]$ lies completely in D. Set $\mathbf{h} = \mathbf{x} - \mathbf{a}$. Then there is a $\theta \in (0, 1)$ such that

$$f(\mathbf{x}) = T_{n,a}(\mathbf{x}) + \frac{1}{(n+1)!} \partial_{\mathbf{h}}^{n+1} f(\mathbf{a} + \theta \mathbf{h}).$$

Proof: Apply Taylor’s theorem in 1D and the chain rule to the function $\phi : [0, 1] \longrightarrow \mathbb{R}$ given by

$$\phi(t) := f(\mathbf{a} + t\mathbf{h}).$$