TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computer Science

Final test Advanced Calculus (2DBN10), Thursday February 1, 2018, 18:00-21:00.

A table of Laplace transforms is part of the problem sheet. No other materials (books, notes, computers, calculators etc.) are allowed.

The answers to the problems have to be formulated and motivated clearly.

1. Let $(y_1, y_2) : [0, \infty) \longrightarrow \mathbb{R}^2$ be the solution of the initial value problem

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

a) Calculate the Laplace transforms Y_1, Y_2 of y_1 and y_2. (3 points)

b) Calculate y_1 and y_2. (3 points)

2. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a twice dfferentiable function such that

$$\partial_x f(x, y) = xy, \qquad \partial_y f(x, y) = ax^2 + bxy$$

for $(x, y) \in \mathbb{R}^2$ with $a, b \in \mathbb{R}$. Find the values of a and b. (2 points)

3. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a strictly increasing, twice continuously differentiable function with second order Taylor polynomial $T_{2,g}$ around 0 given by

$$T_{2,g}(u) = 1 + u + u^2.$$

Let further $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by

$$f(x, y) = g(x - e^{2y}).$$

a) Sketch some level lines for f. (2 points)

b) Give an equation for the tangent to the level line of f in $(0, 0)$. (2 points)

c) Find the second order Taylor polynomial of f rond $(1, 0)$. (3 points)

4. Let $D = \{(x, y) \,|\, 0 \leq x \leq \pi/2,\ 0 \leq y \leq \pi/2\}$. Let $f : D \longrightarrow \mathbb{R}$ be given by

$$f(x, y) = \sin(x) + \sin(y) + \sin(x + y).$$

Find the global maximal value of f on D. (5 points)

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5. Find the maximal value of the function f given by

$$f(x, y, z) = \frac{x + y}{z}$$

on the sphere with equation

$$x^2 + y^2 + (z - 2)^2 = 1.$$

(5 points)

6. Consider the systen of equations

$$x^2 + y^2 + u^2 + v^2 = 4,$$
$$x + y^2 + u^3 + v^4 = 4.$$

Show: In a neighborhood of the point

$$(x_0, y_0, u_0, v_0) = (1, 1, 1, 1),$$

this system can be solved in the form $x = \phi(u, v)$, $y = \psi(u, v)$. Calculate the partial derivatives $\partial_u \phi$, $\partial_u \psi$, $\partial_v \phi$, $\partial_v \psi$ in the point $(u, v) = (1, 1)$. (5 points)

7. Let $K \subset \mathbb{R}^3$ be the pyramid with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(0, 0, 1)$. Calculate

$$\iiint_K x \, dV.$$

(5 points)

8. Let K be the (unbounded) domain in $\mathbb{R}^3$ given by

$$K = \{(x, y, z) \,|\, x^2 + y^2 \leq 1, \ z \geq 0.\}$$

Calculate the (improper) integral

$$\iiint_K z e^{-(x^2 + y^2 + z^2)} \, dV.$$

(5 points)

The grade is determined by dividing the number of points by 4 and rounding to one decimal.

Some standard Laplace transforms and calculation rules:

(schematically, $n \in \mathbb{N}_+$, $\omega \neq 0$, $a > 0$, $b \in \mathbb{R}$)

$F(s) = \mathcal{L}[f](s)$	$f(t)$
$sF(s) - f(0)$	$f'(t)$
$s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$	$f^{(n)}(t)$
$\frac{F(s)}{s}$	$\int_0^t f(\tau)\,d\tau$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(s-b)$	$e^{bt} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{s^2+\omega^2}$	$\frac{1}{\omega} \sin(\omega t)$
$\frac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\frac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3} (\sin(\omega t) - \omega t \cos(\omega t))$
$\frac{s}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega} t \sin(\omega t)$