TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computer Science

Final test Advanced Calculus (2DBN10), Thursday November 1, 2018, 9:00-12:00.

A table of Laplace transforms is part of the problem sheet. No other materials (books, notes, computers, calculators etc.) are allowed.

The answers to the problems have to be formulated and motivated clearly.

1. **a)** Does the linear differential equation

$$y^{(4)}(t) + y(t) = 0$$

have periodic solutions (other than the trivial one $y \equiv 0$)? Give reasons for your answer. (3 points)

b) Find a particular solution for the inhomogeneous linear differential equation

$$u^{(4)}(t) + u(t) = te^t.$$

(2 points)

2. Let $(y_1, y_2) : [0, \infty) \longrightarrow \mathbb{R}^2$ be the solution of the initial value problem

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} e^t \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

a) Calculate the Laplace transforms Y_1, Y_2 of y_1 and y_2. (3 points)

b) Calculate y_1 and y_2. (3 points)

3. Let $D = \{(x, y) \,|\, x \neq 0\}$ and let $f : D \longrightarrow \mathbb{R}$ be given by

$$f(x, y) = \frac{1}{x(y^2 + 1)}, \qquad (x, y) \in D.$$

a) Sketch some level curves for f. (2 points)

b) Give an equation for the tangent plane to the graph of f in the point $(1, 1, 1/2)$. (1 point)

c) Give an equation for the line tangent to the level curves of f through the point $(1, 1)$. (1 point)

d) Give the second order Taylor polynomial for f around $(1, 1)$. (2 points)

4. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be differentiable. Let $g, h : (0, \infty) \longrightarrow \mathbb{R}$ be given by

$$g(t) = f(t, 1/t), \qquad h(t) = f(t, \sqrt{t}).$$

Suppose $g'(1) = 3$, $h'(1) = 0$. Find $\nabla f(1, 1)$.

(**Hint:** Use the chain rule.) (3 points)

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5. Let $D := \{(x,y)\,|\,x^2+y^2 \leq 1\}$ and let $f : D \longrightarrow \mathbb{R}$ be given by

$$f(x,y) = 4x^3 + 4xy^2 + x^2, \qquad (x,y) \in D.$$

Find the global maximum and the global minimum of f on D. (5 points)

6. Find the global maximum and the global minimum for the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by

$$f(x,y,z) = xz - z + \frac{y^2}{2}$$

on the ellipsoid given by the equation

$$x^2 + \frac{y^2}{2} + z^2 = 1.$$

(5 points)

7. **a)** Show that there are a neighborhood D of $(0,0)$ in $\mathbb{R}^2$ and a differentiable function $\phi : D \longrightarrow \mathbb{R}$ such that $\phi(0,0) = 0$ and

$$\phi(x,y) + xe^{y\phi(x,y)} = 0$$

for all $(x,y) \in D$. (2 points)

b) Calculate the linearization of ϕ around $(0,0)$, and calculate $\phi_{xx}(0,0)$. (3 points)

8. Find the volume of the body K in $\mathbb{R}^3$ given by

$$K = \{(x,y,z)\,|\,x^2+y^2+z^2 \leq 4, \quad z \geq 1, \quad z \geq \sqrt{x^2+y^2}\}.$$

(5 points)

The grade is determined by dividing the number of points by 4 and rounding to one decimal.

Some standard Laplace transforms and calculation rules:

(schematically, $n \in \mathbb{N}_+$, $\omega \neq 0$, $a > 0$, $b \in \mathbb{R}$)

$F(s) = \mathcal{L}[f](s)$	$f(t)$
$sF(s) - f(0)$	$f'(t)$
$s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$	$f^{(n)}(t)$
$\dfrac{F(s)}{s}$	$\int_0^t f(\tau)\, d\tau$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(s-b)$	$e^{bt} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{s^2+\omega^2}$	$\frac{1}{\omega} \sin(\omega t)$
$\dfrac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3}(\sin(\omega t) - \omega t \cos(\omega t))$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega} t \sin(\omega t)$