EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

Final test Advanced Calculus (2DBN10), Thursday January 31, 2019, 18:00-21:00.

A table of Laplace transforms is part of the problem sheet. No other materials (books, notes, computers, calculators etc.) are allowed.

The answers to the problems have to be formulated and motivated clearly.

1. **a)** Find all solutions of the differential equation

$$y^{(3)}(t) - y''(t) + 2y(t) = 0$$

that satisfy $y(t) \to 0$ as $t \to -\infty$. (2 POINTS)

b) Find a particular solution for the inhomogeneous linear differential equation

$$u^{(3)}(t) - u''(t) + 2u(t) = \cos t.$$

(2 POINTS)

2. Let $(y_1, y_2) : [0, \infty) \longrightarrow \mathbb{R}^2$ be the solution of the initial value problem

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

a) Calculate the Laplace transforms Y_1, Y_2 of y_1 and y_2. (3 POINTS)

b) Calculate y_1 and y_2. (3 POINTS)

3. Let the function $f : \mathbb{R} \times (\mathbb{R} \setminus \{1\}) \to \mathbb{R}$ be given by

$$f(x, y) = \frac{e^{-x} y}{1 - y}.$$

a) Sketch the level curves of f corresponding to the values $-2, -1, 0, 1$, and 2. (2 POINTS)

b) Give an equation for the tangent plane to the graph of f in the point $(0, 2, -2)$. (1 POINT)

c) Give an equation for the line tangent to the level line passing throught the point $(0, 2)$ in this point. (1 POINT)

d) Give the third order Taylor polynomial for f around the point $(0, 0)$. (2 POINTS)

please turn over

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4. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be differentiable and assume

$$\frac{d}{dt} f(t,t)|_{t=1} = 1, \quad \frac{d}{dt} f(t,t^2)|_{t=1} = 1/2.$$

For all $\alpha \in \mathbb{R}$, calculate $\frac{d}{dt} f(t,t^\alpha)|_{t=1}$. (3 points)

5. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by

$$f(x,y) = (x^2 - y^2)e^{-(x^2+y^2)}$$

Find all critical points of f and determine their type. (5 points)

6. **a)** Let two points $Q_1 = (a_1, b_1, c_1), Q_2 = (a_2, b_2, c_2) \in \mathbb{R}^3$ be given and assume that $a_1, a_2 > 0$. Find the points $P = (x, y, z)$ on the unit sphere (given by $x^2+y^2+z^2 = 1$) for which

$$f(x,y,z) = |PQ_1|^2 + |PQ_2|^2$$

takes the maximal and minimal value. Here $|PQ_k|$ denotes the distance from P to Q_k, $k = 1, 2$. (5 points)

b) Generalize the result to $n \geq 2$ points $Q_k = (a_k, b_k, c_k)$, $k = 1, \ldots, n$. (No derivation of the result is demanded here.) (1 point)

7. **a)** Show that there are an interval I around 2 and two differentiable functions $\phi, \psi : I \longrightarrow \mathbb{R}$ such that $\phi(2) = 0$, $\psi(2) = 4$, and

$$\begin{aligned} \phi(x)^2 + \sin(\phi(x)\psi(x)) &= 2 - x, \\ \phi(x) + \psi(x) &= x^2 \end{aligned}$$

for all $x \in I$. (3 points)

b) Calculate the linearizations of ϕ and ψ around 2. (2 points)

8. Let K be the body in $\mathbb{R}^3$ given by

$$K = \{(x,y,z) \,|\, x^2 + y^2 \leq 1,\ 0 \leq z \leq x + 1\}.$$

Calculate

$$\iiint_K z \, dV.$$

(5 points)

The grade is determined by dividing the number of points by 4 and rounding to one decimal.

Some standard Laplace transforms and calculation rules:

(schematically, $n \in \mathbb{N}_+$, $\omega \neq 0$, $a > 0$, $b \in \mathbb{R}$)

$F(s) = \mathcal{L}[f](s)$	$f(t)$
$sF(s) - f(0)$	$f'(t)$
$s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$	$f^{(n)}(t)$
$\dfrac{F(s)}{s}$	$\int_0^t f(\tau)\, d\tau$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(s-b)$	$e^{bt} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{s^2+\omega^2}$	$\frac{1}{\omega} \sin(\omega t)$
$\dfrac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3} (\sin(\omega t) - \omega t \cos(\omega t))$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega} t \sin(\omega t)$