EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

Final test Advanced Calculus (2DBN10), Thursday October 31, 2019, 9:00-12:00.

A table of Laplace transforms is part of the problem sheet. No other materials (books, notes, computers, calculators etc.) are allowed.

The answers to the problems have to be formulated and motivated clearly.

1. Find the general real solution to the system of differential equations — 4 pt

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

2. **a)** Find the inverse Laplace transform of the function $F : (0, \infty) \longrightarrow \mathbb{R}$ given by — 3 pt

$$F(s) = \frac{s}{s^2 - 4s + 8} + \frac{e^{-s}}{s^2}.$$

b) Let $f, g : [0, \infty) \longrightarrow \mathbb{R}$ be bounded and continuous functions. Let $h : [0, \infty) \longrightarrow \mathbb{R}$ be given by — 3 pt

$$h(t) := \int_0^t f(\tau) g(t - \tau)\, d\tau.$$

Let $F, G, H : (0, \infty) \longrightarrow \mathbb{R}$ be the Laplace transforms of f, g, and h, respectively. Verify that

$$H(s) = F(s)G(s).$$

3. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(x, y) = (x^2 + 1)(y^2 - 1).$$

a) Find all critical points of f and their type. — 2 pt

b) Sketch the level sets of f for the function values $-2, -1, 0, 1, 2$. — 3 pt

4. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be given by — 5 pt

$$g(z) = (z - 2)^{2019} \arctan(z) + (z - 2)^2 - (z - 2).$$

Let $f : \mathbb{R} \times (\mathbb{R} \setminus \{-1\}) \longrightarrow \mathbb{R}$ be given by

$$f(x, y) = g\left(e^x + \frac{1}{1 + y}\right).$$

Find the second order Taylor polynomial for f around $(x_0, y_0) = (0, 0)$.

5. Consider the system of equations — 5 pt

$$\begin{aligned} x + y + t &= 3, \\ x^2 + z^2 + t^2 &= 3, \\ y^3 + z^3 + t^3 &= 3 \end{aligned}$$

Show that in a neighborhood of the point $(x_0, y_0, z_0, t_0) = (1, 1, 1, 1)$ this system has a unique solution for x, y, z in terms of t, i.e. in terms of functions $x = \xi(t)$, $y = \eta(t)$, $z = \zeta(t)$. Find the linearizations of ξ, η, ζ at $t = 1$.

6. Find the maximal and minimal values of the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by 5 pt

$$f(x, y, z) = x + y + z$$

on the ellipsoid in $\mathbb{R}^3$ with equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1, \qquad a, b, c > 0.$$

Simplify the result as far as possible.

7. Let $D := \{(u, v, w) \in \mathbb{R}^3 \mid u \in [0, 2\pi),\ v \in [0, 2\pi), w \in [0, 1]\}$. Consider the mapping $\Phi : D \longrightarrow \mathbb{R}^3$ given by 5 pt

$$\Phi(u, v, w) = \begin{pmatrix} (2 + w\cos v)\cos u \\ (2 + w\cos v)\sin u \\ w\sin v \end{pmatrix}, \qquad (u, v, w) \in D.$$

This mapping is injective. (No proof demanded.)

Find the volume of $\Phi(D)$.

8. Let K be the body in $\mathbb{R}^3$ given by 5 pt

$$K := \{(x, y, z) \mid z \le x^2 + y^2,\ x^2 + y^2 + (z - 1)^2 \le 1\}.$$

Find the volume of K.

The grade is determined by dividing the number of points by 4 and rounding to one decimal.

Some standard Laplace transforms and calculation rules:

(schematically, $n \in \mathbb{N}_+$, $\omega \neq 0$, $a > 0$, $b \in \mathbb{R}$)

$F(s) = \mathcal{L}[f](s)$	$f(t)$
$sF(s) - f(0)$	$f'(t)$
$s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$	$f^{(n)}(t)$
$\dfrac{F(s)}{s}$	$\int_0^t f(\tau)\, d\tau$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(s-b)$	$e^{bt} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{s^2+\omega^2}$	$\frac{1}{\omega} \sin(\omega t)$
$\dfrac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3}(\sin(\omega t) - \omega t \cos(\omega t))$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega} t \sin(\omega t)$