EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics and Computer Science

Final test Advanced Calculus (2DBN10), Thursday January 30, 2020, 18:00-21:00.

A table of Laplace transforms is part of the problem sheet. No other materials (books, notes, computers, calculators etc.) are allowed.

The answers to the problems have to be formulated and motivated clearly.

1. **a)** Find the general solution to the differential equation 4 pt

$$y'''(t) + y''(t) + 2y'(t) + 2y(t) = e^{-t}.$$

b) Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be a continuous function such that $f(t) \equiv 0$ for $t \geq 1$. 2 pt
Let $u : [0, \infty) \longrightarrow \mathbb{R}$ be a solution to the differential equation

$$u'''(t) + u''(t) + 2u'(t) + 2u(t) = f(t).$$

Show that u is bounded on the interval $[1, \infty)$.

2. **a)** Find the inverse Laplace transform of the function $F : (0, \infty) \longrightarrow \mathbb{R}$ given by 3 pt

$$F(s) = \frac{s^2 + 4}{s^3 + s^2 - 2}.$$

b) Let $g : [0, \infty) \longrightarrow \mathbb{R}$ be a bounded and continuous function. Let G denote its Laplace transform. 2 pt
Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be given by

$$f(t) := \int_0^{2t} e^{-\tau} g(\tau)\, d\tau.$$

Express the Laplace transform of f in terms of G.

3. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(x, y) = x^2 + \sin y.$$

a) Sketch the level sets of f for the function values $0, 1,$ and 2. 3 pt

b) Give an equation for the tangent line to the level curve of f that passes through the point $(1, \pi/4)$. 1 pt

4. Find the minimal and maximal value of the function f given by 5 pt

$$f(x, y) := \frac{x + 2y}{1 + x^2 + 4y^2}$$

on the (closed) domain $D := \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + 4y^2 \leq 2\}$.

5. Find the minimal and maximal value of the function f given by 5 pt

$$f(x, y, z) := \frac{x + y}{z - 2}$$

on the surface S given by

$$S := \{(x, y, z) \in \mathbb{R}^3 \,|\, 2x^2 + y^2 + z^2 = 1\}.$$

6. Consider the map $(x, y) \mapsto (u, v)$ given by the equations

$$\begin{aligned} u &= e^x + e^y, \\ v &= \sin x + 2\sin y. \end{aligned}$$

a) Show that near $(x_0, y_0) = (0, 0)$ this map is invertible, i.e. the above system is locally uniquely solvable for $x = \xi(u, v)$, $y = \eta(u, v)$, with differentiable functions ξ, η defined near $(u_0, v_0) = (2, 0)$. 3 pt

b) Calculate 2 pt

$$\frac{\partial(\xi, \eta)}{\partial(u, v)}(2, 0).$$

7. Let D be the domain in $\mathbb{R}^2$ given by $D := \{(x, y) \,|\, x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 1\}$. Consider the mapping $\Phi : D \longrightarrow \mathbb{R}^2$ given by 5 pt

$$\Phi(x, y) := \begin{pmatrix} x^3 - 3xy^2 \\ 3x^2y - y^3 \end{pmatrix}, \qquad (x, y) \in D.$$

This mapping is injective. (No proof demanded.)

Find the area of $\Phi(D)$.

8. Let K be the body in $\mathbb{R}^3$ given by 5 pt

$$K := \{(x, y, z) \,|0 \leq x \leq 1,\ 0 \leq y \leq 1,\ z \geq 0,\ z \leq x,\ z \leq y^2.\}.$$

Find the volume of K.

The grade is determined by dividing the number of points by 4 and rounding to one decimal.

Some standard Laplace transforms and calculation rules:

(schematically, $n \in \mathbb{N}_+$, $\omega \neq 0$, $a > 0$, $b \in \mathbb{R}$)

$F(s) = \mathcal{L}[f](s)$	$f(t)$
$sF(s) - f(0)$	$f'(t)$
$s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$	$f^{(n)}(t)$
$\frac{F(s)}{s}$	$\int_0^t f(\tau)\, d\tau$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(s-b)$	$e^{bt} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{s^2+\omega^2}$	$\frac{1}{\omega} \sin(\omega t)$
$\frac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\frac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3} (\sin(\omega t) - \omega t \cos(\omega t))$
$\frac{s}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega} t \sin(\omega t)$