EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

Final test Advanced Calculus (2DBN10), Thursday October 29, 2020, 13:30-.

A table of Laplace transforms is part of the problem sheet. No other materials (books, notes, computers, calculators etc.) are allowed.

The answers to the problems have to be formulated and motivated clearly.

PART I:

1. Find the general solution of the ODE (5 pt)

$$y^{(3)}(t) + \dot{y}(t) - 2y = e^t, \qquad t \in \mathbb{R}.$$

2. For the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by (3 pt)

$$f(x, y) = y(x^2 - y^2), \quad (x, y) \in \mathbb{R}^2,$$

sketch the level curves for the function values -1, 0, and 1. Indicate clearly which curve corresponds to which of these function values.

3. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be twice continuously differentiable. Let the functions $g, h : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$g(t) = f(t, t), \qquad h(t) = f(\sqrt{2}\cos t, \sqrt{2}\sin t), \qquad t \in \mathbb{R}.$$

a) Assume that $g'(1) = 3$, $h'(\pi/4) = 1$. Calculate $\nabla f(1, 1)$. (3 pt)

b) Assume additionally $g''(1) = 0$, $h''(\pi/4) = 1$. Calculate $\partial_{12} f(1, 1)$. (3 pt)

4. Find the global maximum and minimum of the function f given by (6 pt)

$$f(x, y) = \frac{2x + y}{1 + 4x^2 + y^2}$$

on the domain $D = \{(x, y) \,|\, 4x^2 + y^2 \leq 4\}$.

PART II:

5. Find the solution $(y_1, y_2) = (y_1(t), y_2(t))$ to the ODE system (6 pt)

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -11 & -18 \\ 6 & 10 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad t \in \mathbb{R},$$

for which we have $y_1(0) = 1$ and $y_1(t), y_2(t) \to 0$ as $t \to \infty$.

6. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth (sufficiently often differentiable) function such that its second order Taylor polynomial $T = T_{f,2,(1,0)}$ around $(1, 0)$ is given by (4 pt)

$$T(u, v) = u - 1 + 2v - (u - 1)^2 + (u - 1)v - 2v^2.$$

Find the second order Taylor polynomial around $(0, 0)$ for the function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$g(x, y) = f\left(e^x, \frac{y}{1 - y}\right), \qquad (x, y) \in \mathbb{R}^2.$$

7. Find the global maximum and minimum of the function f given by 6 pt

$$f(x,y,z) = \frac{x+y}{z+2}$$

on the unit sphere $S := \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 1\}$.

8. Let K be the body in $\mathbb{R}^3$ given by 4 pt

$$K := \{(x,y,z) \,|\, x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 2,\ z \leq 1\}.$$

Calculate the integral

$$\iiint_K z \, dV.$$

The grade is determined by dividing the number of points by 4 and rounding to one decimal.

Some standard Laplace transforms and calculation rules:

(schematically, $n \in \mathbb{N}_+$, $\omega \neq 0$, $a > 0$, $b \in \mathbb{R}$)

$F(s) = \mathcal{L}[f](s)$	$f(t)$
$sF(s) - f(0)$	$f'(t)$
$s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$	$f^{(n)}(t)$
$\dfrac{F(s)}{s}$	$\int_0^t f(\tau)\, d\tau$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(s-b)$	$e^{bt} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{s^2+\omega^2}$	$\frac{1}{\omega} \sin(\omega t)$
$\dfrac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3} (\sin(\omega t) - \omega t \cos(\omega t))$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega} t \sin(\omega t)$