EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics and Computer Science

FINAL TEST ADVANCED CALCULUS (2DBN10), THURSDAY JANUARY 28, 2021, 18:00-21:15.

A table of Laplace transforms is part of the problem sheet. No other materials (books, notes, computers, calculators etc.) are allowed.

The answers to the problems have to be formulated and motivated clearly.

PART A:

1. Let $f_1, f_2 : [0,\infty) \longrightarrow \mathbb{R}$ be bounded and continuous functions with Laplace transforms F_1 and F_2, respectively. Let $a_1, a_2 \in \mathbb{R}$ and let y_1, y_2 be the solution of the initial value problem

$$\begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \quad t \in \mathbb{R}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

a) Find the Laplace transforms of y_1 and y_2 in terms of $F_{1,2}$ and $a_{1,2}$. 2 pt

b) In the special case $f_1(t) = e^{-t}$, $f_2(t) = -e^{-t}$, $a_1 = 1$, $a_2 = 0$, find y_1 and y_2. 3 pt

2. Let $D := \{(x,y) \in \mathbb{R}^2 \,|\, |y| \neq 1\}$ and $f : D \longrightarrow \mathbb{R}$ be given by

$$f(x,y) = \frac{x^2}{1-y^2}, \quad (x,y) \in D.$$

a) Sketch the level lines of f corresponding to the function values $-2,-1,0,1,2$. 3 pt

b) Give an equation for the tangent plane to the graph of f in the point $(1,2,-1/3)$. 1 pt

c) Give an equation for the tangent line to the level line passing through the point $(1,2)$. 1 pt

3. Let $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by

$$\Phi(x,y) = \begin{pmatrix} x^3 - y^3 \\ xy \end{pmatrix}, \qquad (x,y) \in \mathbb{R}^2.$$

a) Show that Φ is locally invertible near $(1,1)$, i.e., the system of equation 2 pt

$$\Phi(x,y) = (u,v)^\top$$

is uniquely solvable in a neighborhood of $(x_0,y_0,u_0,v_0) = (1,1,0,1)$ as

$$(x,y)^\top = \Phi^{-1}(u,v).$$

b) Calculate the Jacobian $D(\Phi^{-1})(0,1)$. 3 pt

4. For $a > 0$, define the tetrahedron 5 pt

$$T_a : \{(x,y,z) \,|\, x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq a\}.$$

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be continuous and have the properties

$$\int_{T_1} f(x,y,z)\, dV = 1, \quad f(\lambda x, \lambda y, \lambda z) = \lambda^2 f(x,y,z) \text{ for all } \lambda > 0.$$

For arbitrary $a > 0$, find

$$\int_{T_a} f(x,y,z)\, dV.$$

Hint: Use the change-of-variables theorem with an appropriate mapping from T_1 to T_a.

PART B:

5. Let $y : \mathbb{R} \longrightarrow \mathbb{R}$ be a solution to the differential equation 5 pt

$$\frac{d^{2021}y}{dt^{2021}} - y = 0.$$

Show that there is a constant $C > 0$ such that

$$|y(t)| \leq Ce^t \quad \text{for all } t \geq 0.$$

6. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a differentiable function. Let $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by 4 pt

$$\Phi(x, y) = f(x^2 + y^3, xy).$$

Assume $\nabla\Phi(1, 1) = (1, 2)^\top$. Find $\nabla f(2, 1)$.

7. Let a, b, c be three fixed positive numbers. Find the minimal value of the function f given by 6 pt

$$f(x, y, z) = xyz$$

on the surface S given by

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1, \quad x > 0,\ y > 0,\ z > 0 \right\}.$$

8. Let K be the body in $\mathbb{R}^3$ given by 5 pt

$$K := \{(x, y, z) \,|\, x^2 + y^2 + z^2 \leq 1,\ x^2 + y^2 + (z - 1)^2 \leq 1.\}.$$

Calculate the integral

$$\iiint_K z \, dV.$$

The grade is determined by dividing the number of points by 4 and rounding to one decimal.

Some standard Laplace transforms and calculation rules:

(schematically, $n \in \mathbb{N}_+$, $\omega \neq 0$, $a > 0$, $b \in \mathbb{R}$)

$F(s) = \mathcal{L}[f](s)$	$f(t)$
$sF(s) - f(0)$	$f'(t)$
$s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$	$f^{(n)}(t)$
$\dfrac{F(s)}{s}$	$\int_0^t f(\tau)\, d\tau$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(s-b)$	$e^{bt} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{s^2+\omega^2}$	$\frac{1}{\omega} \sin(\omega t)$
$\dfrac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3} (\sin(\omega t) - \omega t \cos(\omega t))$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega} t \sin(\omega t)$