EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

Final test Advanced Calculus (2DBN10), Thursday November 4, 2021, 13:30-16:30.

A table of Laplace transforms is part of the problem sheet. No other materials (books, notes, computers, calculators etc.) are allowed.

The answers to the problems have to be formulated and motivated clearly.

1. Find the general solution of the ODE 5 pt

$$y'''(t) - y(t) = e^t.$$

2. **a)** Let $\underline{u} = (u_1, u_2) : (0, \infty) \longrightarrow \mathbb{R}^2$ and $\underline{v} = (v_1, v_2) : (0, \infty) \longrightarrow \mathbb{R}^2$ be two solutions to the ODE system 3 pt

$$\begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} 1/t \\ \ln t \end{pmatrix}, \qquad t > 0.$$

Show that $|\underline{u}(t) - \underline{v}(t)|_{\mathbb{R}^2} \to 0$ as $t \to \infty$.

Hint: Do not try to find explicit representations for $\underline{u}$ and $\underline{v}$.

b) Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be bounded and continuous, with Laplace transform $F : (0, \infty) \longrightarrow \mathbb{R}$. Let $g : [0, \infty) \longrightarrow \mathbb{R}$ be given by 3 pt

$$g(t) := \int_t^{2t} f(\tau)\, d\tau, \qquad t \geq 0.$$

Express the Laplace transform G of g in terms of F.

3. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by

$$f(x, y) = y^2 - yx^2.$$

a) Sketch the level curves of f corresponding to the function values $-2, -1, 0, 1, 2$. Indicate clearly which curves correspond to the different values. 3 pt

b) Give an equation for the plane in $\mathbb{R}^3$ tangent to the graph of f in the point $(1, 2, 2)$. 1 pt

c) Give an equation for the line tangent to the level curve passing through the point $(1, 2)$. 1 pt

4. Give the third-order Taylor approximation for the function f given by 4 pt

$$f(x, y) = \sin\left(\frac{y}{2 - x}\right)$$

around the point $(x_0, y_0) = (1, 0)$.

5. Find the maximal and minimal value of the function f given by $f(x, y) = x^2 + 2y^2$ on the domain D given by 5 pt

$$D = \{(x, y) \in \mathbb{R}^2 \,|\, x > 0,\ y > 0,\ x + y \leq 3,\ xy \geq 2\}.$$

please turn over

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6. Let S be the surface in $\mathbb{R}^3$ given by 5 pt

$$S := \{(x, y, z) \,|\, x > 0,\ y > 0,\ z > 0,\ xyz = 1\}.$$

Let $a > 0$, $b > 0$, $c > 0$ be fixed. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be given by

$$f(x, y, z) = ax + by + cz.$$

Find the minimal value of f on S. Simplify the result as far as possible.

7. Let $\Phi : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2$ be given by 5 pt

$$\Phi(u, v) = \begin{pmatrix} \frac{2u}{u^2+v^2} \\ \frac{-v}{u^2+v^2} \end{pmatrix}$$

This mapping is injective. (No proof of this fact is demanded.) Let D be the domain in $\mathbb{R}^2$ given by

$$D := \{(x, y) \,|\, 1 \leq x^2 + y^2 \leq 4.\}$$

Calculate the area of the image $\Phi(D)$.

8. Let K be the the body in $\mathbb{R}^3$ given by 5 pt

$$K = \{(x, y, z) \,|\, \sqrt{x^2 + y^2} + |z - 1| \leq 1\}.$$

Calculate the integral

$$\iiint_K z \, dV.$$

The grade is determined by dividing the number of points by 4 and rounding to one decimal.

Some standard Laplace transforms and calculation rules:

(schematically, $n \in \mathbb{N}_+$, $\omega \neq 0$, $a > 0$, $b \in \mathbb{R}$)

$F(s) = \mathcal{L}[f](s)$	$f(t)$
$sF(s) - f(0)$	$f'(t)$
$s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$	$f^{(n)}(t)$
$\dfrac{F(s)}{s}$	$\int_0^t f(\tau)\, d\tau$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(s-b)$	$e^{bt} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{s^2+\omega^2}$	$\frac{1}{\omega} \sin(\omega t)$
$\dfrac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3} (\sin(\omega t) - \omega t \cos(\omega t))$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega} t \sin(\omega t)$