EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

Final test Advanced Calculus (2DBN10), Thursday January 27 2022, 18:00-21:00 .

A table of Laplace transforms is part of the problem sheet. No other materials (books, notes, computers, calculators etc.) are allowed.

The answers to the problems have to be formulated and motivated clearly.

1. Consider the ODE system 5 pt

$$\dot{y} = \begin{pmatrix} 1 & 5 \\ -1 & a \end{pmatrix} y,$$

where $a \in \mathbb{R}$ is a parameter. For which value(s) of a does this system have periodic solutions (other than the trivial solution $y \equiv 0$)? For these value(s) of a, give the general solution.

2. Find a particular solution to the ODE 6 pt

$$y''(t) + y(t) = \frac{1}{\cos t}, \qquad t \in (-\pi/2, \pi/2).$$

3. Let $D := \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}$ and $f : D \longrightarrow \mathbb{R}$ be given by

$$f(x, y) = \frac{4x^2}{y} + y, \qquad (x, y) \in D.$$

a) Sketch the level curves of f corresponding to the function values $-2, -1, 1, 2$. Describe the level curves in geometric terms. 3 pt

b) Give an equation for the plane in $\mathbb{R}^3$ tangent to the graph of f in the point $(1, 1, 5)$. 1 pt

c) Give an equation for the line tangent to the level curve passing through the point $(1, 1)$. 1 pt

4. Let $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be differentiable with $\nabla u(0, 0) = (1, 2)^\top$. For $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ we define $v := u \circ F$, i.e. $v : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is given by 3 pt

$$v(x, y) = u(F_1(x, y), F_2(x, y)), \qquad (x, y) \in \mathbb{R}^2.$$

Give an example of a function $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that v is differentiable and

$$\nabla v(0, 0) = (3, 1)^\top.$$

please turn over

FINAL TEST ADVANCED CALCULUS (2DBN10),
THURSDAY JANUARY 27 2022, 18:00-21:00 .

---

5. Let $D := \{(x,y) \in \mathbb{R}^2 \,|\, x \neq 0, y \neq 0, x - y \neq 4\}$ and let $f : D \longrightarrow \mathbb{R}$ be given by 5 pt

$$f(x,y) = \frac{1}{x} - \frac{4}{y} + \frac{9}{4 - x + y}.$$

Find all critical points of f and determine their types.

6. Let L be the intersection curve of the the surfaces S_1 and S_2 in $\mathbb{R}^3$, given by 5 pt

$$S_1 := \{(x,y,z) \in \mathbb{R}^3 \,|\, z(x^2+y^2) = 1\}, \quad S_2 := \{\in \mathbb{R}^3 \,|\, xyz^2 = 1,\ x > 0, y > 0, z > 0\}.$$

Find the point on L which is closest to the (x,y)-plane.

7. **a)** Show that on some open interval I around 0, there are differentiable functions $\xi, \eta : I \longrightarrow \mathbb{R}$ that satisfy the equations 3 pt

$$\left.\begin{aligned} \xi(t) &= t + e^{t\eta(t)}, \\ \eta(t) &= t + e^{2t\xi(t)}, \end{aligned}\right\} \qquad t \in I.$$

Hint: Apply the Implicit Function theorem to a suitable system of equations.

b) Find the derivatives $\xi'(0)$, $\eta'(0)$, $\xi''(0)$. 3 pt

.

8. Let K be the the body in $\mathbb{R}^3$ given by 5 pt

$$K = \{(x,y,z) \,|0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le x^2,\ 0 \le z \le y^2\}.$$

Calculate the integral

$$\iiint_K x \, dV.$$

The grade is determined by dividing the number of points by 4 and rounding to one decimal.

Some standard Laplace transforms and calculation rules:

(schematically, $n \in \mathbb{N}_+$, $\omega \neq 0$, $a > 0$, $b \in \mathbb{R}$)

$F(s) = \mathcal{L}[f](s)$	$f(t)$
$sF(s) - f(0)$	$f'(t)$
$s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$	$f^{(n)}(t)$
$\dfrac{F(s)}{s}$	$\int_0^t f(\tau)\, d\tau$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(s-b)$	$e^{bt} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{s^2+\omega^2}$	$\frac{1}{\omega} \sin(\omega t)$
$\dfrac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3} (\sin(\omega t) - \omega t \cos(\omega t))$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega} t \sin(\omega t)$