EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

Final test Advanced Calculus (2DBN10), March 2022, 18:00-21:00 .

A table of Laplace transforms is part of the problem sheet. No other materials (books, notes, computers, calculators etc.) are allowed.

The answers to the problems have to be formulated and motivated clearly.

1. Consider the ODE 4 pt

$$y'''(t) + 5y''(t) + 9y'(t) = 0, \qquad .$$

Does it have solutions y that satisfy $y(t) \to \infty$ as $t \to \infty$? Give reasons for your answer.

2. **a)** Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be given by 2 pt

$$f(t) := \begin{cases} 1 & \text{if } t \in [0, 1], \\ 0 & \text{if } t \in (1, \infty). \end{cases}$$

Give the Laplace transform of f.

b) Find the solution to the initial value problem 4 pt

$$y''(t) + 4y(t) = f(t), \quad y(0) = 1, y'(0) = 0, \quad t \geq 0.$$

3. Let $D := \{(x, y) \in \mathbb{R}^2 \mid y \neq -1\}$ and let $f : D \longrightarrow \mathbb{R}$ be given by

$$f(x, y) = \frac{xy}{y + 1}, \qquad (x, y) \in D.$$

a) Sketch the level lines of f for the values $-2, -1, 0, 1, 2$. Indicate the values to which the lines correspond. 3 pt

b) Give an equation for the plane tangent to the graph of f in the point $(-1, 1, -1/2)$. 1 pt

c) Give an equation for the line tangent to the elvel curve passing through the point $(-1, 1)$. 1 pt

4. Give the second order Taylor polynomial around the point $(0, 0)$ for the function f given by 4 pt

$$f(x, y) = \frac{1}{1 + x + e^y}.$$

5. Let D be the triangle in $\mathbb{R}$ with the corners $(0, 0)$, $(0, 1)$, and $(2, 0)$, including the boundary. Let $f : D \longrightarrow \mathbb{R}$ be given by 5 pt

$$f(x, y) = x^2y + 2xy^2 - 2xy, \qquad x \in D.$$

Find the maximal and minimal value of f in D.

6. Let $S := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Let $f : S \longrightarrow \mathbb{R}$ be given by 6 pt

$$f(x, y, z) = x(y + 2z), \qquad (x, y, z) \in S.$$

Find the maximal value for f, and all points (x, y, z) at which this value is taken.

7. **a)** Show that in a neighborhood of the point $(x_0, y_0, u_0, v_0) = (1, -1, 0, 0)$, the system of equations 2 pt

$$x^2 + y^2 + u - v = 2,$$
$$x^3 + y^3 + u^2 + v^2 = 0$$

can be solved in the form $x = \xi(u, v)$, $y = \eta(u, v)$ with differentiable functions ξ and η.

b) Calculate the partial derivatives of ξ and η with respect to u and v in the point $(u_0, v_0) = (0, 0)$. 3 pt

8. Let K be the body in $\mathbb{R}^3$ given by 5 pt

$$K := \{(x, y, z) \,|\, (x^2 + y^2)z^2 \geq 4,\ \sqrt{x^2 + y^2} + z \leq 3,\ z > 0.\}$$

Calculate the volume of K.

The grade is determined by dividing the number of points by 4 and rounding to one decimal.

Some standard Laplace transforms and calculation rules:

(schematically, $n \in \mathbb{N}_+$, $\omega \neq 0$, $a > 0$, $b \in \mathbb{R}$)

$F(s) = \mathcal{L}[f](s)$	$f(t)$
$sF(s) - f(0)$	$f'(t)$
$s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$	$f^{(n)}(t)$
$\dfrac{F(s)}{s}$	$\int_0^t f(\tau)\, d\tau$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(s-b)$	$e^{bt} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{s^2 + \omega^2}$	$\frac{1}{\omega} \sin(\omega t)$
$\dfrac{s}{s^2 + \omega^2}$	$\cos(\omega t)$
$\dfrac{1}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega^3} (\sin(\omega t) - \omega t \cos(\omega t))$
$\dfrac{s}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega} t \sin(\omega t)$