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Department of Mathematics and Computer Science

Final test Advanced Calculus (2DBN10), November 3 2022, 13:30-16:30 .

A table of Laplace transforms is part of the problem sheet. No other materials (books, notes, computers, calculators etc.) are allowed.

The answers to the problems have to be formulated and motivated clearly.

1. Find the real general solution to the homogenous linear ODE system — 5 pt

$$\dot{y} = \begin{pmatrix} -1 & 1 \\ -5 & 3 \end{pmatrix} y, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

2. **a)** Find the general solution to the inhomogenous linear ODE — 3 pt

$$\ddot{y} + 3\dot{y} + 2y = e^{-t}.$$

b) Let u be the solution to the initial value problem — 2 pt

$$\ddot{u} + 3\dot{u} + 2u = f, \quad u(0) = 1, u'(0) = -4,$$

where $f : [0, \infty) \longrightarrow \mathbb{R}$ is given by

$$f(t) = \begin{cases} \sqrt{2 - t^2} & \text{if } t \in [0, 1), \\ e^{1-t} & \text{if } t \geq 1. \end{cases}$$

Show that $u(t) \to 0$ as $t \to +\infty$.
Hint: Do not try to calculate $u(t)$ explicitly!

3. Let $D = \{(x, y) \in \mathbb{R}^2 \mid x \neq y\}$ and let $f : D \longrightarrow \mathbb{R}$ be given by

$$f(x, y) = \frac{x^2 + y}{x - y} \qquad (x \neq y).$$

a) Sketch the level curves for this function that correspond to the function values $-2, -1, 0, 1, 2$. Clearly indicate which curves belongs to which value. — 3 pt

b) Give the linearization of f in the point $(0, -1)$. — 1 pt

4. Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a differentiable function. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by — 5 pt

$$f(u, v) := g(u^2 - v^2, 2uv).$$

Assume further that

$$\nabla f(\tfrac{1}{2}, \tfrac{1}{2}) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Calculate $\nabla g(0, \frac{1}{2})$.

5. Find all critical points of the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by — 5 pt

$$f(x, y) = x^2 y + y^3 - y$$

and determine their type.

6. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be given by 6 pt

$$f(x, y, z) = \frac{1}{3}x^3 + y^2 + z.$$

Find the minimal and maximal value that f takes on the sphere with equation

$$x^2 + y^2 + z^2 = 2.$$

7. Let $E \subset \mathbb{R}^2$ be a domain with area 3. Let 5 pt

$$D := \{(x + y, x - y) \mid (x, y) \in E\}.$$

Find the area of D.

Hint: Apply the Change-of-Variable theorem.

8. Let K be the the body in $\mathbb{R}^3$ given by 5 pt

$$K = \{(x, y, z) \mid x \geq 0,\ y \geq 0,\ z \geq 0,\ x + y + z \leq 3,\ xy \geq 2\}.$$

Find the volume of K.

The grade is determined by dividing the number of points by 4 and rounding to one decimal.

Some standard Laplace transforms and calculation rules:

(schematically, $n \in \mathbb{N}_+$, $\omega \neq 0$, $a > 0$, $b \in \mathbb{R}$)

$F(s) = \mathcal{L}[f](s)$	$f(t)$
$sF(s) - f(0)$	$f'(t)$
$s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$	$f^{(n)}(t)$
$\frac{F(s)}{s}$	$\int_0^t f(\tau)\, d\tau$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(s-b)$	$e^{bt} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{s^2+\omega^2}$	$\frac{1}{\omega} \sin(\omega t)$
$\frac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\frac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3} (\sin(\omega t) - \omega t \cos(\omega t))$
$\frac{s}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega} t \sin(\omega t)$