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Department of Mathematics and Computer Science

Final test Advanced Calculus (2DBN10), January 26, 2023, 18:00–21:00 .

A table of Laplace transforms is part of the problem sheet. No other materials (books, notes, computers, calculators etc.) are allowed.

The answers to the problems have to be formulated and motivated clearly.

1. Consider the homogenous linear ODE system (4 pt)

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}-3 & a\\ -2 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \tag{1}$$

with a parameter $a \in \mathbb{R}$.

For which values of this parameter is it true that all solution to (1) converge to 0 as $t \to +\infty$?

(Consider all possible values of a, and give reasons for your answer. You may disregard cases in which the coefficient matrix is not diagonalizable.)

2. **a)** Find the inverse Laplace transform of the function F given by (4 pt)

$$F(s) = \frac{s^2 + 13s + 15}{s^3 + 4s^2 + 5s}.$$

b) Let f be a bounded and continuous function on $[0, \infty)$ with Laplace transform F. Let $G : (0, \infty) \longrightarrow \mathbb{R}$ be defined by (2 pt)

$$G(s) = \frac{F(s/2)}{s}.$$

Express the inverse Laplace transform g of G in terms of f.

3. Let $D := \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}$. Let the function $f : D \longrightarrow \mathbb{R}$ be defined by

$$f(x, y) = x^2 + \frac{1}{y}, \qquad (x, y) \in D.$$

a) Sketch the level curves of f corresponding to the function values -2, -1, 0, 1, and 2. Indicate clearly the function values to which the individual curves correspond. (3 pt)

b) Give the linearization of f in the point $(1, -2)$. (1 pt)

c) Give an equation for the line tangent to the level curve passing through the point $(1, -2)$. (1 pt)

4. Give the second order Taylor approximation around the point $(1, 1)$ for the function f given by (4 pt)

$$f(x, y) = e^{\frac{x}{y}}.$$

5. Let $D := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$ and let $f : D \longrightarrow \mathbb{R}$ be defined by (6 pt)

$$f(x, y) = x^2 + y^2 + x^2 y.$$

Find the global maximum and minimum of f on D.

6. Let $\ell \in \mathbb{R}^3$ be the intersection curve of the plane with equation $x + y + z = 1$ and the paraboloid with equation $z = x^2 + y^2$. Find the points on ℓ having the largest and smallest z-coordinate. 5 pt

7. **a)** Show that there is an interval I around $t_0 = 0$ and two differentiable functions $\xi, \eta : I \longrightarrow \mathbb{R}$ such that $x = \xi(t)$ and $y = \eta(t)$ solve the system of equation 3 pt

$$\begin{aligned} \sin x + 2\sin y &= t, \\ e^x + e^y &= t^2 + 2. \end{aligned}$$

(**Hint:** Apply the Implicit Function theorem around a suitably chosen point $(x_0, y_0, 0)$.)

b) Find $\xi'(0)$ and $\eta'(0)$. 2 pt

8. Let K be the body in $\mathbb{R}^3$ given by 5 pt

$$K := \{(x, y, z) \mid z \geq 0,\ x^2 + y^2 \leq z^4 \leq 1\}.$$

Calculate

$$\iiint_K z\, dV.$$

The grade is determined by dividing the number of points by 4 and rounding to one decimal.

Some standard Laplace transforms and calculation rules:

(schematically, $n \in \mathbb{N}_+$, $\omega \neq 0$, $a > 0$, $b \in \mathbb{R}$)

$F(s) = \mathcal{L}[f](s)$	$f(t)$
$sF(s) - f(0)$	$f'(t)$
$s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$	$f^{(n)}(t)$
$\dfrac{F(s)}{s}$	$\int_0^t f(\tau)\,d\tau$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(s-b)$	$e^{bt} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{s^2 + \omega^2}$	$\frac{1}{\omega} \sin(\omega t)$
$\dfrac{s}{s^2 + \omega^2}$	$\cos(\omega t)$
$\dfrac{1}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega^3}(\sin(\omega t) - \omega t \cos(\omega t))$
$\dfrac{s}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega} t \sin(\omega t)$