EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

Final test Advanced Calculus (2DBN10), November 2, 2023, 13:30–16:30.

A table of Laplace transforms is part of the problem sheet. No other materials (books, notes, computers, calculators etc.) are allowed.

The answers to the problems have to be formulated and motivated clearly.

1. **a)** Consider the initial value problem for the homogenous linear ODE system (3 pt)

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 11 & -12 \\ 8 & -9 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{1}$$

Find all initial values $(z_1, z_2)^\top$ such that the solution to (1) converges to 0 as $t \to +\infty$.

b) Find a particular solution to the inhomogenous linear ODE system (4 pt)

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 11 & -12 \\ 8 & -9 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} e^t \\ 1 \end{pmatrix}.$$

2. **a)** Find the inverse Laplace transform of the function F given by (3 pt)

$$F(s) = \frac{1 - e^{-s/2}}{s^2 + 2s + 3}.$$

b) Let f be a bounded, continuously differentiable function on $[0, \infty)$ such that its derivative is also bounded. Let F denote the Laplace transform of f. Show that (2 pt)

$$\lim_{s \to \infty} sF(s) = f(0).$$

3. Let $D := \{(x, y) \in \mathbb{R}^2 \mid y \neq x^2\}$. Let the function $f : D \longrightarrow \mathbb{R}$ be defined by

$$f(x, y) = \frac{x}{x^2 - y}, \qquad (x, y) \in D.$$

a) Sketch the level curves of f corresponding to the function values -2, -1, 0, 1, and 2. Indicate clearly the function values to which the individual curves correspond. (3 pt)

b) Give an equation for the plane tangent to the graph of f in the point $(2, 1, 2/3)$. (1 pt)

c) Give an equation for the line tangent to the level curve passing through the point $(2, 1)$. (1 pt)

4. Give the second order Taylor approximation around the point $(1, 0)$ for the function f given by (4 pt)

$$f(x, y) = \frac{1}{2 - x\cos(y)}.$$

5. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by

$$f(x, y) = x^4 + y^4 - 8xy.$$

a) Find all critical points of f and determine their types. 4 pt

b) Does f have a global maximum? (Motivate your answer.) 1 pt

6. Find the minimal and maximal value of the function f given by 5 pt

$$f(x, y) = e^x(x^2 - y^2)$$

on the curve with equation

$$(x + 2)^2 + y^2 = 4.$$

(**Hint:** The inequality $e^{-2+\sqrt{2}} < (1+\sqrt{2})/4$ is helpful here. No proof of it is demanded.)

7. Let D be the triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Let $\Phi : D \longrightarrow \mathbb{R}^2$ be given by 4 pt

$$\Phi(x, y) = \begin{pmatrix} x^2 - y^2 \\ xy \end{pmatrix}, \qquad (x, y) \in D.$$

The function Φ is injective. (No proof of this is demanded.)

Find the area of the image $\Phi(D)$.

8. Let K be the body in $\mathbb{R}^3$ given by 5 pt

$$K := \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 4,\ x \geq 1\}.$$

Calculate

$$\iiint_K x \, dV.$$

The grade is determined by dividing the number of points by 4 and rounding to one decimal.

Some standard Laplace transforms and calculation rules:

(schematically, $n \in \mathbb{N}_+$, $\omega \neq 0$, $a > 0$, $b \in \mathbb{R}$)

$F(s) = \mathcal{L}[f](s)$	$f(t)$
$sF(s) - f(0)$	$f'(t)$
$s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$	$f^{(n)}(t)$
$\frac{F(s)}{s}$	$\int_0^t f(\tau)\, d\tau$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(s-b)$	$e^{bt} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{s^2+\omega^2}$	$\frac{1}{\omega} \sin(\omega t)$
$\frac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\frac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3} (\sin(\omega t) - \omega t \cos(\omega t))$
$\frac{s}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega} t \sin(\omega t)$