EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

Final test Advanced Calculus (2DBN10), January 25, 2024, 18:00–21:00.

A table of Laplace transforms is part of the problem sheet. No other materials (books, notes, computers, calculators etc.) are allowed.

The answers to the problems have to be formulated and motivated clearly.

1. Consider the homogenous linear ODE system 5 pt

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{1}$$

Find its real general solution. Does the system have any periodic solutions (other than the trivial solution $y \equiv 0$)?

2. Find a particular solution to the third-order linear ODE 5 pt

$$y^{(3)}(t) + 3\ddot{y}(t) - 4y(t) = e^{-2t} + 1.$$

3. Let $D := \{(x, y) \in \mathbb{R}^2 \,|\, x \neq 0,\ y \neq 1\}$. Let the function $f : D \longrightarrow \mathbb{R}$ be defined by

$$f(x, y) = \frac{y}{x(1-y)}, \qquad (x, y) \in D.$$

a) Sketch the level curves of f corresponding to the function values -2, -1, 0, 1, and 2. Indicate clearly the function values to which the individual curves correspond. 3 pt

b) Give an equation for the plane tangent to the graph of f in the point $(1, 2, -2)$. 1 pt

c) Give an equation for the line tangent to the level curve passing through the point $(1, 2)$. 1 pt

4. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be differentiable in $(0, 0)$ with directional derivatives 3 pt

$$D_v f(0, 0) = 3/\sqrt{2}, \quad D_w f(0, 0) = -3/\sqrt{5},$$

where $v := (1, 1)/\sqrt{2}$, $w := (2, -1)/\sqrt{5}$. Calculate $\nabla f(0, 0)$.

5. Let $D \subset \mathbb{R}^2$ be the triangle with vertices $(0,0)$, $(0,2)$, $(2,0)$ (with its boundary included). Let $f : D \longrightarrow \mathbb{R}$ be given by 6 pt

$$f(x,y) = x + y - x^2 y.$$

Find the minimal and maximal value of f on D.

6. Find the minimal and maximal value of the function f given by 5 pt

$$f(x,y,z) = z^2 + xy - y$$

on the unit sphere (i.e. the surface with equation $x^2 + y^2 + z^2 = 1$).

7. Consider the system of equations

$$\begin{aligned} e^{xt} + x + y &= 3, \\ e^{yt} + x^2 + y^3 &= 3. \end{aligned}$$

a) Under which conditions on the point $(t_0, x_0, y_0) \in \mathbb{R}^3$ can this system be uniquely solved for $x = \xi(t)$, $y = \eta(t)$ in a neighborhood of this point? 2 pt

b) Check that the point $(t_0, x_0, y_0) = (0,1,1)$ satisfies these conditions, and calculate the linearizations of ξ and η around $t_0 = 0$. 4 pt

8. Let K be the body in $\mathbb{R}^3$ given by 5 pt

$$K := \{(x,y,z) \,|\, z(1-z) \le \sqrt{x^2+y^2} \le 2z(1-z),\ 0 \le y \le x\}.$$

Calculate the volume of K.

The grade is determined by dividing the number of points by 4 and rounding to one decimal.

Some standard Laplace transforms and calculation rules:

(schematically, $n \in \mathbb{N}_+$, $\omega \neq 0$, $a > 0$, $b \in \mathbb{R}$)

$F(s) = \mathcal{L}[f](s)$	$f(t)$
$sF(s) - f(0)$	$f'(t)$
$s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0)$	$f^{(n)}(t)$
$\dfrac{F(s)}{s}$	$\int_0^t f(\tau)\,d\tau$
$\frac{1}{a} F\left(\frac{s}{a}\right)$	$f(at)$
$F(s-b)$	$e^{bt} f(t)$
$e^{-as} F(s)$	$\begin{cases} f(t-a) & \text{if } t \geq a, \\ 0 & \text{if } t < a \end{cases}$
$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{s^2+\omega^2}$	$\frac{1}{\omega} \sin(\omega t)$
$\dfrac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3}(\sin(\omega t) - \omega t \cos(\omega t))$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega} t \sin(\omega t)$