Solutions final test Advanced Calculus (2DBN10) november 2017

1. **a)** Characteristic polynomial has zeroes $\lambda_1 = -1$, $\lambda_{2,3} = -2 \pm i$. Solution to corresponding homogeneous equation:

$$y_H(t) = C_1 e^{-t} + e^{-2t}(C_2 \cos(t) + C_3 \sin(t)).$$

Ansatz to find a particular solution to the inhomogeneous equation:

$$y_P(t) = Ate^{-t} + Be^t.$$

Inserting this into the equation yields $A = 1/2$, $B = 1/20$. So the general solution to the inhomogeneous equation is

$$y(t) = \frac{1}{2}te^{-t} + \frac{1}{20}e^t + C_1 e^{-t} + e^{-2t}(C_2 \cos(t) + C_3 \sin(t)).$$

b) On the interval $[2017, \infty)$, any solution u satsifies the homogeneous equation

$$u'''(t) + 5u''(t) + 9u'(t) + 5u(t) = 0.$$

So there are constants $C_{1,2,3}$ such that

$$u(t) = C_1 e^{-t} + e^{-2t}(C_2 \cos(t) + C_3 \sin(t),$$

and therefore $\lim_{t\to\infty} u(t) = 0$.

2. **a)** According to the rules for calculating Laplace transforms, we get

$$(s^3 - 3s^2 + 3s - 1)Y(s) - s^2 + 3s - 3 = 0$$

and thus

$$Y(s) = \frac{s^2 - 3s + 3}{s^3 - 3s^2 + 3s - 1}.$$

b) Using partial fraction decomposition,

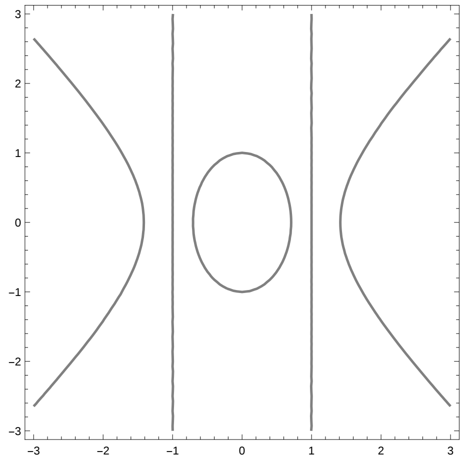
$$Y(s) = \frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3}.$$

By inverse Laplace transform we get

$$y(t) = e^t\left(1 - t + \frac{t^2}{2}\right).$$

3. **a)** We have
$f(x, y) = 1 \Leftrightarrow (x/\sqrt{2})^2 - (y/\sqrt{2})^2 = 1$ (hyperbola),
$f(x, y) = 0 \Leftrightarrow x = \pm 1$,
$f(x, y) = 1/2 \Leftrightarrow (x/(1/\sqrt{2}))^2 + y^2 = 1$ (ellipse)

Sketch:



b) $\nabla f(2,1) = (2,-3/2)^\top$, so an equation is $4(x-2)-3(y-1)=0$.

c) $z = \frac{3}{2} + 2(x-2) - \frac{3}{2}(y-1)$.

4. We have

$$f(x,y,z) = \frac{-\cos(z-\pi)}{1+(x-1)+y}$$
$$= \left(-1 + \frac{(z-\pi)^2}{2} + O((z-\pi)^4)\right)\left(1-((x-1)+y)+((x-1)+y)^2+O(((x-1)^2+y^2)^{3/2})\right)$$

and therefore, by expanding and gathering the terms up to order 2, for the Taylor polynomial

$$T_2(x,y,z) = -1 + (x-1) + y - (x-1)^2 - 2(x-1)y - y^2 + \frac{(z-\pi)^2}{2}.$$

This result can also be obtained using the standard formula and calculating all necessary partial derivatives in $(1,0,\pi)$.

5. We have

$$\nabla f(x,y) = \begin{pmatrix} -\frac{1}{x^2} + \frac{9}{(4-x-y)^2} \\ -\frac{4}{y^2} + \frac{9}{(4-x-y)^2} \end{pmatrix}$$

so $\nabla f(x,y) = 0$ if and only if

$$y^2 = 4x^2 = \frac{4}{9}(4-x-y)^2.$$

We distingush the following cases:

I. $y = 2x$: Then $9x^2 = (4-3x)^2 = 9x^2 - 24x + 16$. So $x_1 = 2/3$ and $y_1 = 4/3$.

II. $y = -2x$: then $9x^2 = (4+x)^2$. This quadratic equation has the solutions $x_2 = -1$ en $x_3 = 2$. The corresponding critical points are $(x_2, y_2) = (-1, 2)$ en $(x_3, y_3) = (2, -4)$.

The second derivative test shows that (x_1, y_1) is a local minimum, and that the two other critical points are saddle points.

6. As f takes positive values for $x > 0$, $y > 0$, $z > 0$, the maximum is positive. If it is taken in any point (x_0, y_0, z_0) in the given ellipsoid, then it is taken in $(|x_0|, |y_0|, |z_0|)$ as well, as this point is also on the ellipsoid and f has the same value there. The maximum is positive, therefore $|x_0| > 0$, $|y_0| > 0$, $|z_0| > 0$.

The Lagrange equations are

$$\begin{pmatrix} y^2 z \\ 2xyz \\ xy^2 \end{pmatrix} + \lambda \begin{pmatrix} 2x \\ 6y \\ 4z \end{pmatrix} = 0$$

with only one solution satisfying $x > 0$, $y > 0$, $z > 0$, namely, $(4, \sqrt{32/3}, \sqrt{8})$. This gives the maximal value $128\sqrt{8}/3$.

(This is a maximum, as $(4, \sqrt{32/3}, \sqrt{8})$ is the only candidate point for a global extremum on the set $x \geq 0$, $y \geq 0$, $z \geq 0$, $x^2 + 3y^2 + 2z^2 = 64$ that satisfies $f > 0$, and f must take its maximum on this set.)[1]

7. Let $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given by

$$F(t, x, y) = \begin{pmatrix} \sin(xt) + x + y \\ \cos(yt) + x + 2y + t \end{pmatrix}.$$

a) The point (t_0, x_0, y_0) has to satisfy the equations: $F(t_0, x_0, y_0) = (0\ 2)^\top$, or

$$\begin{aligned} \sin(x_0 t_0) + x_0 + y_0 &= 0, \\ \cos(y_0 t_0) + x_0 + 2y_0 + t_0 &= 2. \end{aligned}$$

Furthermore, the Jacobi matrix

$$\left(\frac{\partial F}{\partial(x, y)}\right)(t_0, x_0, y_0) = \begin{pmatrix} t_0 \cos(x_0 t_0) + 1 & 1 \\ 1 & -t_0 \sin(y_0 t_0) + 2 \end{pmatrix}$$

has to be regular.

b) We have $F(1, 0, 0) = (0\ 2)^\top$,

$$\left(\frac{\partial F}{\partial(x, y)}\right)(1, 0, 0) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

and this matrix is regular. Therefore, the assumptions of the Implicit Function theorem are satisfied. Furthermore,

$$\begin{aligned} \begin{pmatrix} \phi'(1) \\ \psi'(1) \end{pmatrix} &= -\left(\left(\frac{\partial F}{\partial(x, y)}\right)(1, 0, 0)\right)^{-1} \partial_t F(1, 0, 0) \\ &= -\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \end{pmatrix}. \end{aligned}$$

(This can also be calculated via "implicit differentiation".)

8.

$$Vol(K) = \iiint_K dV = \int_{-1}^{2} \int_{y^2}^{y+2} \int_0^x dz dx dy = 36/5.$$

[1]This argument is not part of the course contents, and therefore not expected as part of the solution.