Oplossingen eindtoets Voortgezette Calculus (2DBN11) februari 2018

Aan deze oplossingen kunnen geen rechten worden ontleend.

1. **a)** According to the properties of the Laplace transform, Y_1, Y_2 are solutions to the linear system

$$\begin{pmatrix} s-2 & 1 \\ -1 & s-2 \end{pmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{s} \end{pmatrix}.$$

This system has the solution

$$Y_1(s) = \frac{s^2-2s-1}{s((s-2)^2+1)}, \quad Y_2(s) = \frac{2s-2}{s((s-2)^2+1)}.$$

b) Partial fraction decomposition and rewriting conveniently:

$$\begin{aligned} Y_1(s) &= \frac{1}{5}\left(-\frac{1}{s} + \frac{6s-14}{(s-2)^2+1}\right) = \frac{1}{5}\left(-\frac{1}{s} + \frac{6(s-2)-2}{(s-2)^2+1}\right), \\ Y_2(s) &= \frac{2}{5}\left(-\frac{1}{s} + \frac{s+1}{(s-2)^2+1}\right) = \frac{2}{5}\left(-\frac{1}{s} + \frac{(s-2)+3}{(s-2)^2+1}\right). \end{aligned}$$

So by inverse Laplace transform

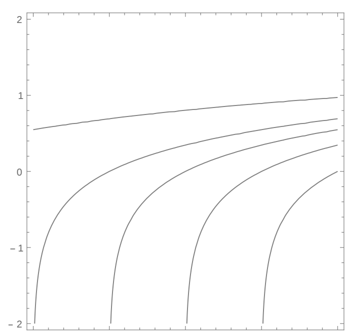
$$y_1(t) = \frac{1}{5}\left(-1 + 6e^{2t}\cos t - 2e^{2t}\sin t\right), \qquad y_2(t) = \frac{2}{5}\left(-1 + e^{2t}\cos t + 3e^{2t}\sin t\right).$$

2. Because of

$$\partial_y \partial_x f(x,y) = x = \partial_x \partial_y f(x,y) = 2ax + by$$

we find $a = \frac{1}{2}$, $b = 0$.

3. **a)** The contour lines are given by $x - e^{2y} = C$, or equivalently $y = \frac{1}{2}\ln(x-C)$.



b) The contour line through $(0,0)$ has equation $y = \frac{1}{2}\ln(x+1)$, so the tangent has equation $y = \frac{x}{2}$.

c) Observe that

$$x - e^{2y} = (x-1) - (e^{2y}-1) = (x-1) - 2y - 2y^2 + O(y^3),$$

so

$$\begin{aligned} f(x,y) &= 1 + (x-1) - 2y - 2y^2 + \left((x-1) - 2y - 2y^2\right)^2 + O(((x-1)^2+y^2)^{3/2}) \\ &= 1 + (x-1) - 2y + (x-1)^2 - 4(x-1)y + 2y^2 + O(((x-1)^2+y^2)^{3/2}). \end{aligned}$$

The second order Taylor polynomial is given by the last expression without the higher order remainder term.
Alternatively, the Taylor polynomial can be found by using $g(0) = 1$, $g'(0) = 1$, $g''(0) = 2$ and the chain rule to calculate all partial derivatives of f up to order 2 in $(1,0)$ and applying the standard formula.

4. Critical points (x,y) have to satisfy $\cos x = \cos y = -\cos(x+y)$, so if $(x,y) \in D$ it follows that $x = y$ and further $\cos x + \cos(2x) = \cos x + 2\cos^2 x - 1 = 0$, so $\cos x = 1/2$ or $\cos x = -1$ where the second possibility is in contradiction with $x \in [0, \pi/2]$. So $x = \pi/3$, and the only critical point of f in D is $(\pi/3, \pi/3)$ with function value $f(\pi/3, \pi/3) = \frac{3}{2}\sqrt{3}$.

Due to the symmetry in x and y, it is sufficient to discuss the boundary components $\{(x,0) \,|\, x \in [0,\pi/2]\}$ and $\{(x,\pi/2) \,|\, x \in [0,\pi/2]\}$. We find

$$f(x,0) = 2\sin x \leq 2 < \frac{3}{2}\sqrt{3},$$

$$f(x,\pi/2) = \sin x + 1 + \cos x \leq \sqrt{2} + 1 < \frac{3}{2}\sqrt{3}.$$

So the global maximum value on D is $\frac{3}{2}\sqrt{3}$.

5. The Lagrange equations are

$$\begin{pmatrix} \frac{1}{z} \\ \frac{1}{z} \\ -\frac{x+y}{z^2} \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2(z-2) \end{pmatrix}.$$

This implies $x = y \neq 0$ and $\lambda = 1/(2xz)$. With this we get further $-2x^2 = z(z-2)$ and from the restriction

$$-z(z-2) + (z-2)^2 = 1,$$

so $z = 3/2$, $x = y = \pm\sqrt{3/8}$. The maximal value is taken in $(\sqrt{3/8}, \sqrt{3/8}, 3/2)$ and is $\sqrt{2/3}$.

6. Let $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be defined by

$$F(x,y,u,v) = \begin{pmatrix} x^2 + y^2 + u^2 + v^2 - 4 \\ x + y^2 + u^3 + v^4 - 4 \end{pmatrix}$$

By the Implicit Function theorem, as F is continuously differentiable, $F(1,1,1,1) = 0$, and

$$D_{(x,y)}F(1,1,1,1) = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$$

is regular, the system $F(x,y,u,v) = (0,0)^\top$ is locally solvable in the form $x = \phi(u,v)$, $y = \psi(u,v)$ near $(1,1,1,1)$. Further

$$\begin{aligned} \begin{pmatrix} \partial_u\phi & \partial_v\phi \\ \partial_u\psi & \partial_v\psi \end{pmatrix}(1,1) &= D_{(u,v)}\begin{pmatrix} \phi \\ \psi \end{pmatrix}(1,1) = -D_{(x,y)}F(1,1,1,1)^{-1}D_{(u,v)}F(1,1,1,1) \\ &= -\begin{pmatrix} 1 & -1 \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} \end{aligned}$$

7.

$$\iiint_K x \, dV = \int_0^1 \int_0^{1-z} \int_0^{1-z} x \, dxdydz = \frac{1}{8}.$$

8. Cylindrical coordinates:

$$\iiint_K ze^{-(x^2+y^2+z^2)} \, dV = \int_0^{2\pi} \int_0^{\infty} \int_0^1 re^{-r^2} ze^{-z^2} \, dV = \frac{\pi}{2}(1 - e^{-1}).$$