Solutions final test Advanced Calculus (2DBN10) november 2018

No rights can be derived from these solutions.

1. **a)** The equation is homogeneous and has charachteristic equation $\lambda^4+1=0$ with the four solutions $\lambda_{1,2}=\frac{1}{\sqrt{2}}(1\pm i)$, $\lambda_{3,4}=\frac{1}{\sqrt{2}}(-1\pm i)$, and therefore the general solution

$$y(t)=e^{t/\sqrt{2}}\big(C_1\cos(t/\sqrt{2})+C_2\sin(t/\sqrt{2})\big)+e^{-t/\sqrt{2}}\big(C_3\cos(t/\sqrt{2})+C_4\sin(t/\sqrt{2})\big).$$

If at least one of the constants C_j is not zero then y is unbounded, hence y is not periodic.

b) Ansatz: $u(t)=(At+B)e^t$. Filling this in yields $2At+2B+4A\equiv t$, so $A=1/2$, $B=-1$, so

$$u(t)=(\tfrac{t}{2}-1)e^t.$$

2. **a)** According to the properties of the Laplace transform, Y_1, Y_2 are solutions to the linear system

$$\begin{pmatrix} s-1 & -3 \\ -3 & s-1 \end{pmatrix}\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix}=\begin{pmatrix} \frac{1}{s-1} \\ 2 \end{pmatrix}.$$

This system has the solution

$$Y_1(s)=\frac{7}{(s-1)^2-9},\quad Y_2(s)=\frac{2(s-1)^2+3}{(s-1)((s-1)^2-9)}.$$

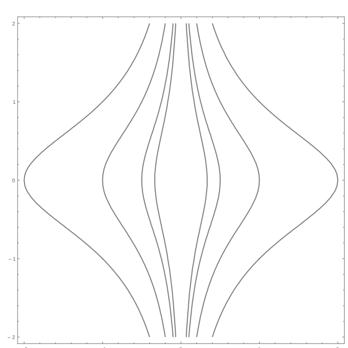
b) Partial fraction decomposition:

$$\begin{aligned} Y_1(s) &= \frac{7}{6}\frac{1}{s-4}-\frac{7}{6}\frac{1}{s+2}, \\ Y_2(s) &= \frac{7}{6}\frac{1}{s-4}-\frac{1}{3}\frac{1}{s-1}+\frac{7}{6}\frac{1}{s+2}. \end{aligned}$$

So by inverse Laplace transform

$$y_1(t)=\tfrac{7}{6}(e^{4t}-e^{-2t}),\qquad y_2(t)=\tfrac{7}{6}(e^{4t}+e^{-2t})-\tfrac{1}{3}e^t.$$

3. **a)** The level curves are given by $f(x,y)=c, c\neq 0$, or equivalently $x=c^{-1}\frac{1}{y^2+1}$.



b) We have $f_x(1,1) = f_y(1,1) = -\frac{1}{2}$, so the tangent plane is given by

$$z = \tfrac{1}{2}(1 - (x-1) - (y-1)).$$

c) The tangent line should run through $(1,1)$ and be orthogonal to $\nabla f(1,1) = -(1/2, 1/2)$, so an equation is

$$(x-1) + (y-1) = 0.$$

d) We can use the 1D Taylor expansions

$$\begin{aligned} \frac{1}{x} &= \frac{1}{1+(x-1)} = 1 - (x-1) + (x-1)^2 + \ldots, \\ \frac{1}{y^2+1} &= \frac{1}{2}\frac{1}{1+\left((y-1)+\frac{(y-1)^2}{2}\right)} = \frac{1}{2}\left(1 - (y-1) + \tfrac{1}{2}(y-1)^2 + \ldots\right) \end{aligned}$$

to find the second order Taylor polynomial

$$p(x,y) = \tfrac{1}{2}(1 - (x-1) - (y-1) + (x-1)^2 + \tfrac{1}{2}(y-1)^2 + (x-1)(y-1)).$$

Alternatively, the Taylor polynomial can be obtained from the standard formula.

4. We have

$$\begin{aligned} g'(1) = &\quad f_x(1,1) - f_y(1,1) = 3, \\ h'(1) = &\quad f_x(1,1) + \tfrac{1}{2} f_y(1,1) = 0. \end{aligned}$$

We can solve this system for the two unknowns $f_x(1,1)$ and $f_y(1,1)$ and get

$$\nabla f(1,1) = (f_x(1,1), f_y(1,1))^\top = (1,-2)^\top.$$

5. Critical points in D are found from solving the equation

$$\nabla f(x,y) = \begin{pmatrix} 12x^2 + 4y^2 + 2x \\ 8xy \end{pmatrix} = 0.$$

From the second equation we get that $x = 0$ or $y = 0$.

1. If $x = 0$ then $y = 0$ from the first equation, $(0,0)$ is a critical point in D with $f(0,0) = 0$.
2. If $x \neq 0$ then $y = 0$ and from the first equation we get $x = -1/6$. Indeed $(-1/6, 0)$ is a critical point in D with $f(-1/6, 0) = 1/108$.

The boundary points satisfy $x^2 + y^2 = 1$, $x \in [-1,1]$, and therefore $f(x,y) = g(x) := 4x + x^2$. It is easy to see that the minimal and maximal value of g on $[-1,1]$ are taken in -1 and 1, respectively, with function values $g(-1) = f(-1,0) = -3$ and $g(1) = f(1,0) = 5$. Comparison with the values in the critical points in D shows that these are the global extrema of f on D.

6. The Lagrange equations are

$$\begin{pmatrix} z \\ y \\ x-1 \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ y \\ 2z \end{pmatrix}, \qquad x^2 + \frac{y^2}{2} + z^2 = 1.$$

From $y = \lambda y$ it follows that $y = 0$ or $\lambda = 1$.

1. If $y = 0$ then $z^2 = 2\lambda x z = x^2 - x$ implies $2x^2 - x = 1$, so $x_1 = -1/2$, $x_2 = 1$. The corresponding points are $(-1/2, 0, \pm\sqrt{3}/2)$ and $(1,0,0)$ with function values

$$f(-1/2, 0, \pm\sqrt{3}/2) = \mp\tfrac{3}{4}\sqrt{3}, \qquad f(1,0,0) = 0.$$

2. If $\lambda = 1$ then $x - 1 = 2z = 4x$, so $x = -1/3$, $z = -2/3$. The corresponding points are $(-1/3, \pm\frac{2}{3}\sqrt{2}, -2/3)$ with function values

$$f(-1/3, \pm\tfrac{2}{3}\sqrt{2}, -2/3) = 4/3.$$

This is the global maximum, while the global minimum is $-\frac{3}{4}\sqrt{3}$.

7. **a)** Define $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$F(x, y, z) = z + xe^{yz}.$$

F is differentiable, $F(0, 0, 0) = 0$, and $\partial_z F(0, 0, 0) = 1 \neq 0$. So the Implicit Function theorem can be applied and yields the statement.

b) The 1D Chain rule gives

$$\begin{aligned}
\phi_x(x, y) + (1 + xy\phi_x(x, y))e^{y\phi(x,y)} &= 0, \\
\phi_y(x, y) + x(\phi(x, y) + y\phi_y(x, y))e^{y\phi(x,y)} &= 0, \\
\phi_{xx}(x, y) + (2y\phi_x(x, y) + xy\phi_{xx}(x, y) + xy^2\phi_x(x, y)^2)e^{y\phi(x,y)} &= 0.
\end{aligned}$$

Filling in $(x, y) = (0, 0)$ yields $\phi_x(0, 0) = -1$, $\phi_y(0, 0) = \phi_{xx}(0, 0) = 0$, and the linearization

$$p(x, y) = -x.$$

8. Spherical coordinates:

$$\begin{aligned}
\iiint_K dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_{1/\cos\phi}^{2} \rho^2 \sin\phi \, d\rho d\phi d\theta = \frac{2\pi}{3}\int_0^{\pi/4} \sin\phi \left(8 - \frac{1}{\cos^3\phi}\right) d\phi \\
&\overset{u=\cos\phi}{=} \frac{2\pi}{3}\int_{\sqrt{2}/2}^{1} (8 - u^{-3})\, du = \pi\left(5 - \frac{8}{3}\sqrt{2}\right)
\end{aligned}$$