Solutions final test Advanced Calculus (2DBN10)
january 2019

No rights can be derived from these solutions.

1. **a)** The equation is homogeneous and has charachteristic equation $\lambda^3 - \lambda^2 + 2 = 0$ with the three solutions $\lambda_1 = -1$, $\lambda_{2,3} = 1 \pm i$, and therefore the general solution

$$y(t) = C_1 e^{-t} + e^t(C_2 \cos(t) + C_3 \sin(t)).$$

If $C_1 \neq 0$ then $|y(t)| \to \infty$ as $t \to -\infty$. On the other hand, all solutions of the form $y(t) = e^t(C_2 \cos(t) + C_3 \sin(t))$ satisfy $y(t) \to 0$ as $t \to -\infty$.

b) Ansatz: $u(t) = A\cos(t) + B\sin(t)$. Filling this in yields $A + 3B = 0$, $3A - B = 1$, so $A = 3/10$, $B = -1/10$, $u(t) = \frac{3}{10}\cos(t) - \frac{1}{10}\sin(t)$.

2. **a)** According to the properties of the Laplace transform, Y_1, Y_2 are solutions to the linear system

$$\begin{pmatrix} s-1 & -1 \\ 1 & s-3 \end{pmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{s-2} \\ \frac{2}{s-2} \end{pmatrix}.$$

This system has the solution

$$Y_1(s) = \frac{s-1}{(s-2)^3}, \quad Y_2(s) = \frac{2s-3}{(s-2)^3}.$$

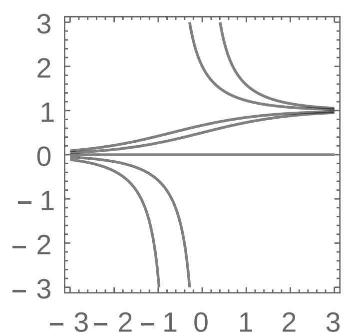
b) Rewrite

$$Y_1(s) = \frac{1}{(s-2)^2} + \frac{1}{(s-2)^3}, \qquad Y_2(s) = \frac{2}{(s-2)^2} + \frac{1}{(s-2)^3}.$$

So by inverse Laplace transform

$$y_1(t) = (t + \tfrac{t^2}{2})e^{2t}, \qquad y_2(t) = (2t + \tfrac{t^2}{2})e^{2t}.$$

3. **a)** The level curves are given by $f(x, y) = c$, or equivalently $y = ce^x/(1 + ce^x)$ for $c \in \{-2, -1, 0, 1, 2\}$.



b) We have $f_x(0,2) = 2,\ f_y(0,2) = 1$, so the tangent plane is given by

$$z = -2 + 2x + (y - 2).$$

c) The tangent line should run through $(0,2)$ and be orthogonal to $\nabla f(0,2) = (2,1)^\top$, so an equation is

$$2x + (y - 2) = 0.$$

d) We can use the 1D Taylor expansions

$$\begin{aligned} e^{-x} &= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + O(x^4), \\ \frac{y}{1-y} &= y + y^2 + y^3 + O(y^4) \end{aligned}$$

to find the third order Taylor polynomial

$$p_3(x,y) = y - xy + y^2 + \frac{1}{2}x^2y - xy^2 + y^3.$$

Alternatively, the Taylor polynomial can be obtained from the standard formula.

4. From the given equations we find by the chain rule

$$\begin{aligned} \partial_1 f(1,1) + \ \partial_2 f(1,1) &= 1, \\ \partial_1 f(1,1) + 2\partial_2 f(1,1) &= 1/2. \end{aligned}$$

We solve this linear system to find $\partial_1 f(1,1) = 3/2$, $\partial_2 f(1,1) = -1/2$ and consequently

$$\frac{d}{dt} f(t, t^\alpha)|_{t=1} = \partial_1 f(1,1) + \alpha \partial_2 f(1,1) = (3 - \alpha)/2.$$

5. The critical points are the zeroes of the gradient given by

$$\nabla f(x,y) = 2 \begin{pmatrix} x(y^2 - x^2 + 1) \\ y(y^2 - x^2 - 1) \end{pmatrix} e^{-(x^2+y^2)}$$

From the first component we get that $x = 0$ or $y^2 - x^2 = -1$. In the first case, the second equation implies $y = 0$ or $y = \pm 1$. In the second case, the second equation implies $y = 0$ and therefore $x = \pm 1$. Summarizing, there are five critical points: $(0,0)$, $(0,\pm 1)$, and $(\pm 1, 0)$.

To determine their type, we calculate the Hessian:

$$\nabla^2 f(x,y) = 2 \begin{pmatrix} -2x^2 - (2x^2 - 1)(y^2 - x^2 + 1) & 2xy(y^2 - x^2) \\ 2xy(y^2 - x^2) & 2y^2 - (2y^2 - 1)(y^2 - x^2 - 1) \end{pmatrix} e^{-(x^2+y^2)}.$$

At the critical points, this is

$$\nabla^2 f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \quad \nabla^2 f(\pm 1, 0) = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} e^{-1}, \quad \nabla^2 f(0, \pm 1) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} e^{-1}.$$

So the second derivative test yields that $(0,0)$ is a saddle point, $(\pm 1, 0)$ are local maxima, and $(0, \pm 1)$ are local minima.

6. **a)** The function f to be minimized is given by

$$f(x,y,z) = \sum [(x - a_i)^2 + (y - b_i)^2 + (z - c_i)^2].$$

Here and in the remaining part of of a), $\sum$ denotes summation over $i = 1, 2$. The Lagrange equations are

$$2\begin{pmatrix} 2x - \sum a_i \\ 2y - \sum b_i \\ 2z - \sum c_i \end{pmatrix} = 2\lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

or

$$(2-\lambda)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sum \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix}.$$

As $a_1 + a_2 > 0$ we have $\lambda \neq 2$ and therefore $(x, y, z) \parallel \sum(a_i, b_i, c_i)$. Consequently, the restriction implies

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \pm \frac{1}{\sqrt{(\sum a_i)^2 + (\sum b_i)^2 + (\sum c_i)^2}} \sum \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix}$$

(In the "+" case the minimum is taken, and in the "-"-case the maximum is taken. A proof of this is not demanded.)

b) The calculations for a) generalize immediately to $n \geq 2$, if all summations are taken from $i = 1$ to n.

7. **a)** Define $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by

$$F(x, y, z) = \begin{pmatrix} x + y^2 + \sin(yz) - 2 \\ -x^2 + y + z \end{pmatrix}.$$

F is differentiable, $F(2, 0, 4) = 0$, and

$$D_{(y,z)}F(x, y, z) = \begin{pmatrix} 2y + z\cos(yz) & y\cos(yz) \\ 1 & 1 \end{pmatrix},$$

in particular

$$D_{(y,z)}F(2, 0, 4) = \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix},$$

so $D_{(y,z)}F(2, 0, 4)$ is regular.

Therefore the Implicit Function theorem can be applied and yields the statement.

b) Differentiating the given equations with respect to x and filling in $x = 2$ yields the linear system

$$\begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \phi'(2) \\ \psi'(2) \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

with solution $\phi'(2) = -1/4$, $\psi'(2) = 17/4$. So the linearizations are

$$\phi(x) \approx -\tfrac{1}{4}(x-2), \qquad \psi(x) \approx 4 + \tfrac{17}{4}(x-2).$$

8. Cylindrical coordinates:

$$\iiint_K z\, dV = \int_0^{2\pi}\int_0^1\int_0^{1+r\cos\theta} zr\, dz dr d\theta = \frac{5}{8}\pi.$$