Solutions final test Advanced Calculus (2DBN10)
October 2019

No rights can be derived from these solutions.

1. The coefficient matrix has the eigenvalues $1 \pm i$ with corresponding eigenvectors $(1, \pm i)^\top$. So the general solution is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1+i)t} + C_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(1-i)t},$$

and in real form

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^t \left(C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \right).$$

2. **a)** Rewrite

$$F(s) = \frac{s-2}{(s-2)^2+4} + \frac{2}{(s-2)^2+4} + \frac{e^{-s}}{s^2}$$

and use the table to find

$$f(t) = \begin{cases} e^{2t}(\cos(2t) + \sin(2t)) & (t \leq 1) \\ e^{2t}(\cos(2t) + \sin(2t)) + t - 1 & (t > 1), \end{cases}$$

where f denotes the inverse Laplace transform of F.

b)

$$H(s) = \int_0^\infty e^{-ts} \int_0^t f(\tau) g(t-\tau)\, d\tau dt = \int_0^\infty \int_0^t e^{-(t-\tau)s} g(t-\tau) e^{-\tau s} f(\tau)\, d\tau dt$$
$$\overset{\theta = t - \tau}{=} \int_0^\infty \int_0^\infty e^{-\theta s} g(\theta) e^{-\tau s} f(\tau)\, d\theta d\tau = G(s)F(s).$$

3. **a)** Critical points satisfy the pair of equations

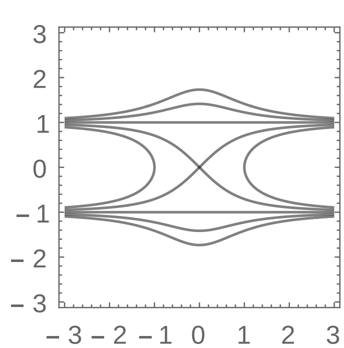
$$\nabla f(x,y) = \begin{pmatrix} 2x(y^2-1) \\ 2y(x^2+1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The second equation implies $y = 0$, and then the first implies $x = 0$, so $(0,0)$ is the only critical point of f. The Hessian is

$$H_f(x,y) = \begin{pmatrix} 2(y^2-1) & 4xy \\ 4xy & 2(x^2+1) \end{pmatrix},$$

so the second-derivative test shows that $(0,0)$ is a saddle point.

b)



4. We obviously have
$$g(z) = -(z-2) + (z-2)^2 + O((z-2)^{2019})$$
and by standard Taylor expansions
$$e^x + \frac{1}{1+y} = 2 + x - y + \frac{x^2}{2} + y^2 + O(|(x,y)|^3)$$
Setting formally $z = e^x + \frac{1}{1+y}$ and collecting the lowest order terms yields
$$f(x,y) = \underbrace{-x + y + \frac{x^2}{2} - 2xy}_{T_2(x,y)} + O(|(x,y)|^3),$$
where T_2 is the second-order Taylor polynomial.

This result can also be obtained by using the standard formula for Taylor polynomials and the chain rule for calculating the partial derivatives of f at $(0,0)$.

5. Let $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be given by
$$F(x,y,z,t) = \begin{pmatrix} x+y+t-3 \\ x^2+z^2+t^2-3 \\ y^3+z^3+t^3-3 \end{pmatrix}.$$
Then
$$D_{(x,y,z)}F(x,y,z,t) = \begin{pmatrix} 1 & 1 & 0 \\ 2x & 0 & 2z \\ 0 & 3y^2 & 3z^2 \end{pmatrix},$$
$F(1,1,1,1) = 0$, and
$$D_{(x,y,z)}F(1,1,1,1) = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 3 & 3 \end{pmatrix}.$$
This matrix is regular, so the Implicit Function theorem yields differentiable functions ξ, η, ζ solving the system near $(1,1,1,1)$. By using the general chain rule or by "implicit differentiation" we find
$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 3 & 3 \end{pmatrix} \begin{pmatrix} \xi'(1) \\ \eta'(1) \\ \zeta'(1) \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0,$$
so $\xi'(1) = \eta'(1) = \zeta'(1) = -\frac{1}{2}$, and the linearization around $t=1$ for all three functions is given by $t \mapsto 1 - \frac{t-1}{2}$.

6. The Lagrange equations are
$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2\lambda \begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \\ \frac{z}{c^2} \end{pmatrix} = 0,$$
so $\lambda \neq 0$ and further
$$\frac{x}{a^2} = \frac{y}{b^2} = \frac{z}{c^2}.$$
Inserting this in the restriction yields
$$\frac{x^2}{a^4}(a^2+b^2+c^2) = 1,$$

hence

$$x = \pm\frac{a^2}{\sqrt{a^2+b^2+c^2}}, \quad y = \pm\frac{b^2}{\sqrt{a^2+b^2+c^2}}, \quad z = \pm\frac{c^2}{\sqrt{a^2+b^2+c^2}},$$

so f has maximal and minimal value

$$\pm\frac{a^2+b^2+c^2}{\sqrt{a^2+b^2+c^2}} = \pm\sqrt{a^2+b^2+c^2}.$$

7. By the change-of-variables theorem,

$$\operatorname{Vol}\Phi(D) = \iiint_{\Phi(D)} 1\,dV = \iiint_D |\det D\Phi|\,dV.$$

Here we have

$$D\Phi(u,v,w) = \begin{pmatrix} -(2+w\cos v)\sin u & -w\sin v\cos u & \cos v\cos u \\ (2+w\cos v)\cos u & -w\sin v\sin u & \cos v\sin u \\ 0 & w\cos v & \sin v \end{pmatrix},$$

so

$$\det D\Phi(u,v,w) = (2+w\cos v)w\begin{vmatrix} -\sin u & -\sin v\cos u & \cos v\cos u \\ \cos u & -\sin v\sin u & \cos v\sin u \\ 0 & \cos v & \sin v \end{vmatrix} = (2+w\cos v)w.$$

As $w\cos v \geq -1$, $w \geq 0$ we have $|\det D\Phi(u,v,w)| = (2+w\cos v)w$ and so

$$\operatorname{Vol}\Phi(D) = \int_0^{2\pi}\int_0^{2\pi}\int_0^1 (2+w\cos v)w\,dwdvdu = 4\pi^2.$$

8. Cylindrical coordinates: The paraboloid $z = r^2$ and the sphere $r^2+(z-1)^2 = 1$ intersect in the circle line $r = z = 1$ (and at the origin). So

$$\operatorname{Vol}(K) = \iiint_K dV = \int_0^{2\pi}\int_0^1\int_{\sqrt{z}}^{\sqrt{1-(z-1)^2}} r\,drdzd\theta = \frac{\pi}{6}.$$