Solutions final test Advanced Calculus (2DBN10)
January 2020

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1. **a)** The characteristic polynomial is $\lambda \mapsto \lambda^3 + \lambda^2\lambda + 2$ with roots $\lambda_1 = -1$, $\lambda_{2,3} = \pm\sqrt{2}i$. So the general solution Y_H of the homogeneous equation is given by

$$y_H(t) = C_1 e^{-t} + C_2 \cos(\sqrt{2}t) + C_3 \sin(\sqrt{2}t).$$

To find a particular solution we use the ansatz $y_P(t) = Ate^{-t}$ and find $A = \frac{1}{3}$. So the general solution is

$$y(t) = C_1 e^{-t} + C_2 \cos(\sqrt{2}t) + C_3 \sin(\sqrt{2}t) + \frac{t}{3}e^{-t}.$$

b) On the interval $[1, \infty)$, u satisfies the homogeneous equation

$$u''' + u'' + 2u' + 2u = 0.$$

Therefore there are constants $C_{1,2,3}$ such that

$$u(t) = C_1 e^{-t} + C_2 \cos(\sqrt{2}t) + C_3 \sin(\sqrt{2}t)$$

there, which is bounded on this interval.

2. **a)** Partial fraction decomposition:

$$F(s) = \frac{1}{s-1} - \frac{2}{s^2+2s+2} = \frac{1}{s-1} - \frac{2}{(s+1)^2+1},$$

so from linearity and the table

$$f(t) = e^t - 2e^{-t}\sin t.$$

b) Define $h, k : [0, \infty) \longrightarrow \mathbb{R}$ by

$$h(\tau) := e^{-\tau} g(\tau), \quad k(t) = \int_0^t h(\tau)\, d\tau,$$

and denote their respective Laplace transforms by H and K. Then

$$H(s) = G(s+1), \quad K(s) = \frac{1}{s} G(s+1),$$

and in view of $f(t) = k(2t)$ we get for the Laplace transform F of f

$$F(s) = \frac{1}{2} K\left(\frac{s}{2}\right) = \frac{1}{s} G\left(\frac{s}{2} + 1\right).$$

3. **a)** **TBD!**

b) The line should be orthogonal to $\nabla f(1, \pi/4) = (2, \sqrt{2}/2)^\top$, so an equation is

$$2(x-1) + \frac{\sqrt{2}}{2}(y - \pi/4) = 0.$$

4. The critical points satisfy

$$\nabla f(x,y) = \frac{1}{(1+x^2+4y^2)^2}\begin{pmatrix} 1+x^2+4y^2-2x(x+2y) \\ 2(1+x^2+4y^2)-8y(x+2y) \end{pmatrix} = 0.$$

This implies

$$1+x^2+4y^2 = 2x(x+2y) = 4y(x+2y).$$

As these terms are positive, we conclude $x+2y \neq 0$ and therefore $x = 2y$. This implies further $1+8y^2 = 16y^2$, so $y = \pm\sqrt{2}/4$, and $(x,y) = \pm(\sqrt{2}/2, \sqrt{2}/4)$. Both points lie in D, the corresponding function values are $\pm\sqrt{2}/2$.

We parameterize the boundary by $t \mapsto (\xi,\eta)(t) := \sqrt{2}\cos t, (\sqrt{2}/2)\sin t$ and find

$$f(\xi(t),\eta(t)) = \frac{\sqrt{2}}{3}(\cos t + \sin t)$$

with maximal and minimal value $\pm 2/3$. So the global maximal and minimal value are $\pm\sqrt{2}/2$.

5. The minimal and maximal value are taken at solutions of the Lagrange equations

$$\frac{1}{z-2}\begin{pmatrix} 1 \\ 1 \\ -\dfrac{x+y}{z-2} \end{pmatrix} = 2\lambda \begin{pmatrix} 2x \\ y \\ z \end{pmatrix}, \qquad 2x^2+y^2+z^2 = 1.$$

For solutions of these equations we have $\lambda \neq 0$, $x \neq 0$, $y \neq 0$, hence $y = 2x$ and $\frac{z}{2x} = -\frac{3x}{z-2}$. So

$$z^2 - 2z = -6x^2 = -2x^2 - y^2 = z^2 - 1,$$

so $z = 1/2$, $(x,y) = \pm(\sqrt{2}/4, \sqrt{2}/2)$, and the maximal and minimal values of f on S are $\pm\sqrt{2}/2$.

6. **a)** $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be given by

$$F(x,y,u,v) = \begin{pmatrix} e^x + e^y - u \\ \sin x + 2\sin y - v \end{pmatrix}.$$

Then $F(x_0,y_0,u_0,v_0) = 0$, and

$$D_{(x,y)}F(x_0,y_0,u_0,v_0) = \begin{pmatrix} e^{x_0} + e^{y_0} \\ \sin x_0 + 2\sin y_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

This matrix is regular, so the statement follows from the Implicit Function Theorem.

b) From $F(\xi(u,v),\eta(u,v),u,v) = 0$ we get from the chain rule

$$D_{(x,y)}F(0,0,2,0)\frac{\partial(\xi,\eta)}{\partial(u,v)}(2,0) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0,$$

so

$$\frac{\partial(\xi,\eta)}{\partial(u,v)}(2,0) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

(This can also be obtained from "implicit differentiation" of both equations separately and solving the resulting linear system for the partial derivatives of ξ and η.)

7. By definition and the change of variables theorem,

$$\text{Area}(\Phi(D)) = \iint_{\Phi(D)} 1\, dA = \iint_D |\det \Phi|\, dA.$$

Further

$$\det D\Phi(x,y) = \det \begin{pmatrix} 3(x^2-y^2) & -6xy \\ 6xy & 3(x^2-y^2) \end{pmatrix} = 9(x^2+y^2)^2.$$

So, using polar coordinates,

$$\text{Area}(\Phi(D)) = 9\iint_D (x^2+y^2)^2\, dxdy = 9\int_0^{\pi/2}\int_0^1 r^5\, drd\theta = 3\pi/4.$$

8.

$$\begin{aligned}\text{Vol}(K) &= \iiint_K dV = \int_0^1\int_0^1\int_0^{\min(x,y^2)} dzdydx \\ &= \int_0^1\int_0^{\sqrt{x}}\int_0^{y^2} dzdydx + \int_0^1\int_{\sqrt{x}}^1\int_0^x dzdydx = 7/30.\end{aligned}$$

Alternative:

$$\text{Vol}(K) = \iiint_K dV = \int_0^1\int_z^1\int_{\sqrt{z}}^1 dydxdz = 7/30.$$